STABILITY THEORY

by

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SYNOPSIS

Stability Theory is the branch of Model Theory which deals with: the sets of formulae known as types, extensions of types, and stable theories, which are special first order theories.
We start off by examining the connection between the concepts of stability and definability. We then investigate the characterization of stability in terms of (order). Most of this thesis is devoted to the concept of forking which is an important concept in stability Theory. We end off by examining the connection between the concepts of ω-stability, superstability and categoricity.

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INTRODUCTION

Model Theory is a branch of Logic which is concerned with the relationship between sets of formulae and relational structures which are described by these formulae. A useful concept in Model Theory is that of a 'type', which is a set of formulae satisfying certain fixed conditions. Stability Theory is the branch of Model Theory which deals with the study of types, special extensions of types and certain theories, known as 'stable' theories, which are defined in terms of types. Stability Theory is largely a tool for other branches of Model Theory.

Stability Theory began with Morley's thesis of 1962, in which he proved that a complete theory in a countable language which is categorical in some uncountable power, is in fact categorical in all uncountable powers. Morley's work led to the introduction of the basic concepts of Stability Theory. The concept of a theory being α-stable was introduced by Rowbottom [64]. ω-stable theories had already been studied by Morley as 'totally transcendental' theories. The more general concept of a theory being stable was due to Shelah [71].

In Chapter 1, we examine the basic results concerning stability. We introduce the concepts of a definable type (due to Shelah [71]) and heirs and co-heirs of types (due to Poizat [77].). The notion of a formula being represented in a type, which we use to define heirs, was introduced by Lascar and Poizat [79]. We prove important results linking these concepts with stability.

In Chapter 2 we investigate the connection between stability and 'no-order'

which was first studied by Pillay, Knivine, Lascan and Shelah. We introduce the 'fundamental order' due to Poizat (Lascan and Poizat [79], Poizat [77]) and the rank $R^p(\psi)$ due to Shelah [78]. The main results of chapter 2 are due to Shelah's work on 'no-order'.

Chapter 3 contains the bulk of this thesis. In this chapter we study the concept of forking. This is one of the most important concepts in stability theory. Forking was introduced by Shelah [78]. We use the definition of forking due to Lascan and Poizat [79] which only applies to stable theories. We also examine the connection between forking and definability. The concepts of 'almost over' and 'definable almost over' studied in this section are due to Shelah [78], while the concepts of a good defining schema and well-definable types are due to Lascan [76].

The last chapter (4.) deals with the connection between ω-stability and categoricity. We also examine the characterization of ω-stability in terms of the concept of superstability which was introduced by Shelah [71]. The main result of this chapter is Morley's Categoricity Theorem.

NOTATION

We use A, B, C, M, N to denote structures and subsets of structures. We do not distinguish between structures and their underlying sets. T is used to denote a theory.

L is used to denote a countable first order language. If A is a subset of a structure for L we use $L(A)$ to denote the expansion of L obtained by adding the names of elements of A to L as new constants. We do not distinguish between names of elements and the elements themselves. $\varphi, \psi, \theta, \sigma$ are used for formulae. x, y, z, v are used for variables and $\underline{x}, \underline{y}, \underline{z}, \underline{v}$ are used for tuples of variables. a, b, c, d, m are used for constants (or elements of structures), and $\underline{a}, \underline{b}$ etc. are used for tuples of constants. $\varphi(\underline{x})$, $\varphi(\underline{x}, \underline{y})$ etc. are used to indicate that the free variables of φ are among $\underline{x}$ or $\underline{x}, \underline{y}$ respectively. If $\varphi(\underline{x}, \underline{y})$ is a formula, then $\varphi(\underline{x}, \underline{a})$ is the formula obtained from $\varphi(\underline{x}, \underline{y})$ by replacing free occurrences of variables in $\underline{y}$ by corresponding constants in $\underline{a}$. $\Sigma, \Delta, \Gamma, \Phi$ are used for sets of formulae. $\Sigma(\underline{x})$ is used to denote the fact that all formulae in Σ have free variables $\underline{x}$ (ie the free variables occur among the variables in $\underline{x}$).

$\alpha, \beta, \gamma, \eta, \delta$ are used to denote cardinals and ordinals. (The context will make it clear whether a cardinal or ordinal is meant.) m, n, k are used for natural numbers while i, j are used to denote elements of an index set (usually natural numbers). We work in ZFC set theory throughout.

We use f, g, h, p to denote functions. ${}^{X}Y$ is used to denote the set of all functions from X to Y. Y^{X} is only used for

exponentiation of cardinals. (e.g. α^β is the cardinal 'α to the power β' while ${}^\beta\alpha$ is the set of functions from β to α.) If $p: X \to Y$ is a function then $p(a)$ indicates the image of an element $a \in X$ under p, while $p[Z]$ denotes the image of a subset $Z \subseteq X$. $p^{-1}[Z]$ denotes the pre-image of a subset $Z \subseteq Y$. This will <u>not</u> apply to defining schemas.

If a set, say X is underlined ("$\underline{X}$"), this indicates that X is a topological space. The symbol $\overline{M}$ is used to indicate a certain fixed model of a theory T as explained in Chapter 1.

We use '$A \subseteq B$' to denote that A is a subset of B while $A \subset B$ denotes that A is a <u>proper</u> subset of B. $A \cup \underline{b}$ is used to denote the union of A and the set of elements in the tuple $\underline{b}$. (As usual $A \leq B$ denotes that A is a substructure of B and $A \prec B$ means that A is an elementary substructure.)

Types and Defining Schemas

1. Definitions and Basic Results:

To begin with we define the notion of a type which is a fundamental concept in Stability Theory.

Let T be a complete theory in a language L. Let $n < \omega$ and let $\underline{x}$ be an n-tuple of variables of L. Let Σ be a set of formulae of L with free variables $\underline{x}$. Σ is called a <u>(complete) n-type</u> of T iff:

a.) $T \cup \Sigma$ is consistent, and

b.) For all formulae $\varphi(\underline{x}) \in L$ either

$$\varphi(\underline{x}) \in \Sigma \quad \text{or} \quad \neg\varphi(\underline{x}) \in \Sigma$$

Σ is called a <u>type</u> of T iff Σ is an n-type of T for some $n < \omega$.

If $n < \omega$ we put $S_n(T) = \{\Sigma : \Sigma \text{ is an } n\text{-type of } T\}$ and we put $S(T) = \bigcup_{n<\omega} S_n(T)$. $S_n(T)$ is thus the set of all n-types of T and $S(T)$ is the set of all types of T.

We are often concerned with theories of particular structures; we thus define the following types:

Let M be a structure and L the appropriate language for M, let $A \subseteq M$, and let $\underline{b} \in M^n$ $(n < \omega)$. Then the <u>type of $\underline{b}$ over A in M</u> is the n-type:

$$tp_M(\underline{b}/A) = \{\varphi(\underline{x}) \in L(A) : M \models \varphi[\underline{b}]\}$$

(where $\underline{x}$ is an n-tuple of variables)

If $A = \emptyset$ we write $tp_M(\underline{b})$ for $tp_M(\underline{b}/A)$.

Thus $tp_M(\underline{b}/A)$ is an n-type of the theory $T = Th(M, a)_{a \in A}$. Also, if T is a theory, then Σ is an n-type of T iff there is a model M of T and an n-tuple $\underline{b}$ in M such that $\Sigma = tp_M(\underline{b})$.

Let M be a structure and $A, B \subseteq M$. Let $\langle b_\beta : \beta < \alpha \rangle$ be an enumeration of B. Let L be the appropriate language for M with its set of variables expanded if necessary so that for all $\beta < \alpha$ there is a variable x_β. We define the <u>'type' of B over A in M</u> to be:

$$tp_M(B/A) = \{\varphi(x_{\beta_0}, \ldots, x_{\beta_{n-1}}) \in L(A) : n < \omega, \beta_0, \ldots, \beta_{n-1} < \alpha \text{ and } M \models \varphi(b_{\beta_0}, \ldots, b_{\beta_{n-1}})\}$$

If Σ is an n-type whose formulae have free variables $\underline{x}$, and Σ' is obtained from Σ by replacing $\underline{x}$ by some other n-tuple of variables, say $\underline{y}$, then we consider Σ and Σ' to be the same type. $tp_M(\underline{b}/A)$ is thus well-defined i.e. the definition does not depend on the choice of $\underline{x}$. Similarly $tp_M(B/A)$ is considered to be unique up to renumbering of variables and elements of B.

It should be noted that $tp_M(B/A)$ is not actually a type when B is infinite.

Let M be a structure and α a cardinal. We say that

M is α-saturated iff for any $A \subseteq M$ with $|A| < \alpha$ we have that every $\Sigma \in S_1(Th(M,a)_{a \in A})$ is realized in M. (ie there is a tuple $\underline{b}$ in M such that for all $\varphi(\underline{x}) \in \Sigma$, $M \models \varphi[\underline{b}]$.)

We mention the following results concerning α-saturation.

<u>Prop 1.1.1</u>

Let M be an α-saturated structure with $\alpha \geq \aleph_0$, and let L be the appropriate language for M, then:

a) If M is infinite, then $|M| \geq \alpha$

b.) If $A \subseteq M$ with $|A| < \alpha$, and Σ is a set of formulae of $L(A)$ in less than α free variables, which is consistent with $Th(M,a)_{a \in A}$, then Σ is realized in M. (we allow the set of variables of $L(A)$ to be expanded to any infinite <u>set</u> of variables.)

c.) If N is a structure such that $N \equiv M$ and $|N| \leq \alpha$, then N is elementary embeddable in M.

d.) If N is an α-saturated structure such that $N \equiv M$ and $|N| = \alpha = |M|$, then $N \cong M$.

e.) (α-homogeneity) If $|M| = \alpha$ and $A \subseteq M$ with $|A| < \alpha$, then any elementary map $f : A \to M$ can be extended to an automorphism $\bar{f} : M \to M$.

<u>Prop 1.1.2</u>

Let T be a complete theory, then for all $\alpha \geq \aleph_0$ T has an α-saturated model.

The above results allow us to make the following conventions:

If T is a complete theory, we choose a cardinal $\alpha > \aleph_0$ to be larger than all sets and structures of interest and (by Prop 1.1.2) we let $\overline{M}$ be an α-saturated model of T. Then by Prop 1.1.1 we may, without loss of generality, assume that all models of T under discussion are elementary submodels of $\overline{M}$. We let L be the (countable) language appropriate for $\overline{M}$ and we assume without mention that the variable set of L can be expanded to any set of variables. From now on T will denote a (countable) complete theory with no finite models and the above conventions and notation will be assumed. We also use the following abbreviations:

If φ is a formula in L, then we write $\models \varphi$ for $\overline{M} \models \varphi$. We write $tp(\underline{b}/A)$ and $tp(B/A)$ for $tp_{\overline{M}}(\underline{b}/A)$ and $tp_{\overline{M}}(B/A)$ respectively, and we write $S_n(A)$ and $S(A)$ for $S_n(Th(\overline{M}, a)_{a \in A})$ and $S(Th(\overline{M}, a)_{a \in A})$ respectively.

We are now ready to define the notion of stability and prove some basic results.

<u>Definition:</u>

Let $\beta \geq \aleph_0$. We say that T is <u>β-stable</u> iff for all $A \subseteq \overline{M}$ with $|A| \leq \beta$ we have $|S_1(A)| \leq \beta$.

We say that T is <u>stable</u> iff there is some $\beta \geq \aleph_0$ such that T is β-stable.

■

We have the following basic results concerning stability:

<u>Prop 1.1.3</u>

T is β-stable iff for all $n \in \mathbb{N}$ and all $A \subseteq \overline{M}$ with $|A| \leq \beta$

we have $|S_n(A)| \leq \beta$.

<u>Proof</u>:

$\Leftarrow$: This follows by choosing $n=1$.

$\Rightarrow$: We prove the result by induction. For $n=1$ this is just the definition of β-stable. Assume the result holds for $n=k$, but that there is an $A \subseteq \overline{M}$ with $|A| \leq \beta$ and $|S_{k+1}(A)| > \beta$. We may assume that all types in $S_{k+1}(A)$ have the same $(k+1)$-tuple of free variables say $\underline{x}$. If $\Sigma \in S_{k+1}(A)$ let Σ^* be obtained from Σ by restricting to β-value in the first k variables of $\underline{x}$. Then $\Sigma^* \in S_k(A)$. By the inductive hypothesis $|S_k(A)| \leq \beta$ so we can find $\{\Sigma_\gamma \in S_{k+1}(A) : \gamma < \beta^+\}$ such that for $\gamma < \eta < \beta^+$ we have $\Sigma_\gamma \neq \Sigma_\eta$ but for all $\gamma < \beta^+$ $\Sigma_\gamma^* = \Delta$ say. Let $c_1, \ldots, c_k \in \overline{M}$ such that for all $\varphi \in \Delta$, $\models \varphi[c_1 \ldots c_k]$. For all $\gamma < \beta^+$ let θ_γ be obtained from Σ_γ by substituting $c_1, \ldots, c_k$ for the first k free variables. Then θ_γ is a type in $S_1(A \cup \{c_1, \ldots, c_k\})$, and $\gamma < \eta < \beta^+$ implies that $\theta_\gamma \neq \theta_\eta$. But by β-stability $|S_1(A \cup \{c_1, \ldots, c_k\})| \leq \beta$ — contradiction. Hence for all $A \subseteq \overline{M}$ $|A| \leq \beta$ implies that $|S_{k+1}(A)| \leq \beta$ and the result follows.

$\blacksquare$

We often refer to $\aleph_0$-stability as ω-stability. To prove an important result concerning ω-stability we introduce the following notation:

If $A \subseteq \overline{M}$ and $\varphi \in L(A)$ we let $[\varphi]_A = \{\Sigma \in S_1(A) : \varphi \in \Sigma\}$
If $n < \omega$ and $f \in {}^n 2$ we let $f+i$ (for $i=0,1$) denote the extension of f to $n+1$ with $f(n) = i$.

Let ${}^{\omega>}\lambda$ denote $\bigcup_{n<\omega} {}^{n}\lambda$.

<u>Prop 1.1.4</u> (Morley [65])

If T is ω-stable then T is β-stable for all $\beta \geq \aleph_0$

<u>Proof:</u>

We prove the result by contraposition, hence assume that T is not β-stable for some $\beta \geq \aleph_0$. Then there is an $A \subseteq \overline{M}$ such that $|A| \leq \beta$ but $|S_1(A)| > \beta$. Let φ be the formula $(x = x)$, Then for all $\Sigma \in S_1(A)$, $\varphi \in \Sigma$, and so $|[\varphi]_A| > \beta$. By induction we can find a set $\{\varphi_f(x) \in L(A) : f \in {}^{\omega>}\lambda\}$ satisfying the following:

i) $\varphi_\emptyset = \varphi$

ii) For all $f \in {}^{\omega>}\lambda$ $|[\varphi_f]_A| > \beta$

iii) For all $f \in {}^{\omega>}\lambda$ and $i = 0, 1$ $\models (\varphi_{f+i} \Rightarrow \varphi_f)$

iv) For all $f \in {}^{\omega>}\lambda$ $\models (\varphi_{f+0} \Rightarrow \neg \varphi_{f+1})$

Let $B = \{a \in A : a \text{ occurs in some } \varphi_f, f \in {}^{\omega>}\lambda\}$. Then $|B| \leq \aleph_0$. For each $g \in {}^{\omega}\lambda$ put $\Sigma_g = \{\varphi_{g|n} : n < \omega\}$, then by (ii) and (iii) any finite subset of Σ_g is consistent. Hence by compactness Σ_g is a consistent set of formulae in $L(B)$. By Zorn's Lemma each Σ_g ($g \in {}^{\omega}\lambda$) can be extended to a type $\Sigma'_g \in S_1(B)$. Then by (iv) $g_1 \neq g_2$ implies $\Sigma'_{g_1} \neq \Sigma'_{g_2}$, hence $|S_1(B)| \geq 2^{\aleph_0}$ and so T is not ω-stable.

∎

When we need to indicate the n-tuple $\underline{x}$ of free variables associated with an n-type Σ we will write $\Sigma(\underline{x})$. We now define the notions of definability and defining schemes. These concepts will allow us to provide a characterization of

stable theories.

<u>Definition:</u>

Let $\Sigma(\underline{x}) \in S(A)$ with $B \subseteq A \subseteq \bar{M}$. We say that Σ is <u>definable over B</u> iff for every formula $\varphi(\underline{x}, \underline{y})$ in L, there is a formula $\psi(\underline{y}) \in L(B)$ such that for any tuple $\underline{a}$ in A we have $\varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})$ iff $\models \psi[\underline{a}]$. If $\Sigma \in S(A)$ we say that Σ is <u>definable</u> iff Σ is definable over A.

Let $\Sigma(\underline{x}) \in S(A)$ be definable over some $B \subseteq A$:
Put $\Delta = \{\varphi(\underline{x}, \underline{y}) \in L : \underline{y}$ a tuple of variables$\}$.
Define a map $d : \Delta \to L(B)$ as follows: If $\varphi(\underline{x}, \underline{y}) \in \Delta$ let $\psi(\underline{y}) \in L(B)$ be as in the definition above; put $d\varphi = \psi$.
Such a map d is called a <u>defining schema over B</u> for $\Sigma(\underline{x})$. If $B = A$ we say that d is a <u>defining schema</u> for $\Sigma(\underline{x})$.

■

If $\Sigma(\underline{x}) \in S(A)$ is of the form $\{\varphi(\underline{x}) \in L : T \vdash (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}))\}$ then $\Sigma(\underline{x})$ is called the <u>principal type</u> generated by $\psi(\underline{x})$.

Let $\Sigma(\underline{x}) \in S(A)$ be principal with generator $\psi(\underline{x})$ and suppose $\psi(\underline{x})$ contains constants $\{a_1, \ldots, a_n\} \subseteq A$. Then for all $\varphi(\underline{x}, \underline{y}) \in L$ put $d\varphi = (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}, \underline{y}))$. Then if $\varphi(\underline{x}, \underline{y}) \in L$, we have for any tuple $\underline{b}$ in A, $\varphi(\underline{x}, \underline{b}) \in \Sigma(\underline{x})$ iff $\models (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}, \underline{b}))$ iff $\models d\varphi[\underline{b}]$. Also each $d\varphi$ is a formula in $L(\{a_1, \ldots, a_n\})$. Hence d is a defining schema over $\{a_1, \ldots, a_n\}$ for $\Sigma(\underline{x})$. Thus any principal type is definable over some finite set.

For an example of a non-principal definable type consider

the case where $\bar{M}$ is an abelian group. Let $b \in \bar{M}$ and let $A \subseteq \bar{M}$. Put $\Sigma(x) = tp(b/A)$. For any formula $\varphi \in L$ there is a formula ψ such that $\models \varphi \Leftrightarrow \psi$, where ψ is Boolean combination of formulae of the form $t(\underline{x}) = 0$ and $(\exists z)(z + z + \cdots + z = y)$, where t is a term in L. (See Eklof and Fisher [72]). These formulae can be written as $kx = s(\underline{y})$ and $(\exists z)(nz = x + s(\underline{y}))$ where $k, n < \omega$. Let φ be the formula $kx = y$ $(k < \omega)$. Put $d\varphi(y)$ equal to $y = a$ if $\models kb = a$, otherwise put $d\varphi(y)$ equal to $\neg(y = y)$. Then we have $(kx = y) \in \Sigma(x)$ iff $\models d\varphi(y)$. Now let φ be the formula $(\exists z)(nz = kx + y)$ $(n, k < \omega)$. If there is an $a \in A$ such that $\models (\exists z)(nz = kb + a)$, put $d\varphi(y)$ equal to $(\exists z)(nz = a - y)$, otherwise put $d\varphi(y)$ equal to $\neg(y = y)$. Then $(\exists z)(nz = kx + y) \in \Sigma(x)$ iff $\models d\varphi(y)$. We define $d\varphi(y)$ similarly if y is replaced by a term $s(\underline{y})$ in φ. Thus we obtain a defining schema d for $\Sigma(x)$, and so $\Sigma(x)$ is definable.

Not all types are definable however. We postpone examples of such types until we have developed some results which will make the explanations of the examples easier.

2. Uniqueness of Defining Schemas up to Logical Equivalence.

If $A \leq \bar{M}$ and $\Sigma \in S(A)$ is definable, let d_1 and d_2 be defining schemas for Σ. Can we conclude that $d_1 = d_2$? The answer is no. This is no loss however since we will show that for all $\varphi(\underline{x}, \underline{y}) \in L$ we have $\models (\forall \underline{y})(d_1\varphi(\underline{y}) \Leftrightarrow d_2\varphi(\underline{y}))$. Thus although defining schemas for types over models are not unique, they are unique up to logical equivalence. (To see that $d_1 \neq d_2$ in general, let d_1 be a defining schema for

some type $\Sigma(\underline{x}) \in S(A)$ and let $\varphi(\underline{x}, \underline{y}) \in L$. Let $\psi(\underline{y})$ be any formula such that $\models (\forall \underline{y})(d_1\varphi(\underline{y}) \Leftrightarrow \psi(\underline{y}))$. Let d_2 be identical to d_1 with the exception that $d_2\varphi(\underline{y}) = \psi(\underline{y})$. Then d_2 is also a defining schema for $\Sigma(\underline{x})$ and if $\psi(\underline{y})$ was chosen to be different from $d_1\varphi(\underline{y})$, e.g. $\psi(\underline{y}) = \neg\neg d_1\varphi(\underline{y})$, then $d_1 \neq d_2$.) Before proving our result concerning the equivalence of defining schemas we consider the possibility of using a defining schema for a type $\Sigma \in S_n(A)$ to construct an extension of Σ in $S_n(B)$ where $B \supseteq A$.

If $A \subseteq B \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S_n(A)$ is a definable type with defining schema d. Put $d(B) = \{\varphi(\underline{x}, \underline{b}) : \varphi(\underline{x}, \underline{y}) \in L,\ \underline{b}$ a tuple in B such that $\models d\varphi[\underline{b}]\}$

Then clearly $\Sigma(\underline{x}) \subseteq d(B)$ but is $d(B)$ an n-type in $S_n(B)$? The next proposition gives a sufficient condition for $d(B)$ to be an n-type.

<u>Prop 1.2.1</u> (Lascar [76])

Let $A \subseteq B \subseteq \overline{M}$ with $A \preceq \overline{M}$. Let $\Sigma(\underline{x}) \in S_n(A)$ be a definable type with defining schema d. Then $d(B) \in S_n(B)$ and $d(B)$ is definable with defining schema d.

<u>Proof:</u>

We need to show that $d(B)$ is complete and consistent with $\overline{M}$. Let $\varphi(\underline{x}, \underline{y}) \in L$. Then if $\underline{a}$ is a tuple in A we have:

$\models d(\neg\varphi)[\underline{a}]$ iff $\neg\varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})$ iff $\models \neg d\varphi[\underline{a}]$. Thus $A \models (\forall \underline{y})(d(\neg\varphi)(\underline{y}) \Leftrightarrow \neg d\varphi(\underline{y}))$ and since $A \prec \overline{M}$ we have $\models (\forall \underline{y})(d(\neg\varphi)(\underline{y}) \Leftrightarrow \neg d\varphi(\underline{y}))$. (Recall that by our conventions $A \preceq \overline{M}$ implies $A \prec \overline{M}$.) Now if $\underline{b}$ is a tuple in B and $\varphi(\underline{x}, \underline{b}) \notin d(B)$ then by definition of $d(B)$ we have $\not\models d\varphi[\underline{b}]$ so $\models \neg d\varphi[\underline{b}]$. Thus $\models d(\neg\varphi)[\underline{b}]$ and so $\neg\varphi(\underline{x}, \underline{b}) \in d(B)$.

Thus $d(B)$ is complete.

For consistency: we have for any $\varphi_1(\underline{x}, \underline{y}_1), \ldots, \varphi_k(\underline{x}, \underline{y}_k)$ in L, $A \models (\forall \underline{y}_1, \ldots, \underline{y}_k)(d\varphi_1(\underline{y}_1) \wedge \ldots \wedge d\varphi_k(\underline{y}_k) \Rightarrow (\exists \underline{x})$
$(\varphi_1(\underline{x}, \underline{y}_1) \wedge \ldots \wedge \varphi_k(\underline{x}, \underline{y}_k)))$
and hence $\overline{M}$ satisfies this formula. Now if for $i = 1, \ldots, k$ we have $\varphi_i(\underline{x}, \underline{b}_i) \in d(B)$ for some tuples $\underline{b}_i$ in B, then for $i = 1, \ldots, k$ $\models d\varphi_i[\underline{b}_i]$ and so $\models d\varphi_1[\underline{b}_1] \wedge \ldots \wedge d\varphi_k[\underline{b}_k]$. Hence $\models (\exists \underline{x})(\varphi_1(\underline{x}, \underline{b}_1) \wedge \ldots \wedge \varphi_k(\underline{x}, \underline{b}_k))$ and so $\{\varphi_1(\underline{x}, \underline{b}_1), \ldots, \varphi_k(\underline{x}, \underline{b}_k)\}$ is consistent with $\overline{M}$. Hence by compactness we have that $d(B)$ is consistent with $\overline{M}$.

Thus $d(B) \in S_n(B)$ and by definition of $d(B)$, d is then a defining scheme for $d(B)$.

$\blacksquare$

We now prove our result about the uniqueness of defining schemes up to logical equivalence.

<u>Thm 1.2.2</u>

Let $A \preceq \overline{M}$ and let $\Sigma(\underline{x}) \in S(A)$ be definable with defining schemes d_1 and d_2. Then for all $\varphi(\underline{x}, \underline{y}) \in L$ we have $\models (\forall \underline{y})(d_1\varphi(\underline{y}) \Leftrightarrow d_2\varphi(\underline{y}))$. Also if $A \preceq B \preceq \overline{M}$ then the types $d_1(B)$ and $d_2(B)$ are equal.

<u>Proof:</u>

Let $\varphi(\underline{x}, \underline{y}) \in L$. Then for all tuples $\underline{a}$ in A we have: $\models d_1\varphi[\underline{a}]$ iff $\varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})$ iff $\models d_2\varphi[\underline{a}]$, and hence $\models (d_1\varphi[\underline{a}] \Leftrightarrow d_2\varphi[\underline{a}])$. Thus $A \models (\forall \underline{y})(d_1\varphi(\underline{y}) \Leftrightarrow d_2\varphi(\underline{y}))$, and since $A \prec \overline{M}$ we have $\models (\forall \underline{y})(d_1\varphi(\underline{y}) \Leftrightarrow d_2\varphi(\underline{y}))$.

It then follows from the definition of $d_1(B)$ and $d_2(B)$ that they

are equal. ($d_1(B)$ and $d_2(B)$ are types by Prop 1.2.1.).

It should be noted that in Prop 1.2.1 and Thm 1.2.2 it is required that A is a submodel of $\overline{M}$. If A is just a subset of $\overline{M}$ then the results do not hold. For example, let L be a language with a single binary predicate symbol R. Let T be the theory that says that R is an equivalence relation with three infinite classes. Let $A \subseteq \overline{M}$ be a block of R and let $a \in A$. Then the formula $\neg R(x,a)$ axiomatizes a type $\Sigma(x) \in S_1(A)$. We give two defining schemas for $\Sigma(x)$. It suffices to define the schemas for formulae of the form $R(x,y)$, $\neg R(x,y)$, $x=y$ and $\neg(x=y)$. (since by elimination of quantifiers we then get a schema defined on all formulae.) Define d_1 and d_2 by:

$$d_1 R(x,y) = \neg(y=y) \qquad d_2 R(x,y) = \neg R(y,a)$$
$$d_1 \neg R(x,y) = R(y,a) \qquad d_2 \neg R(x,y) = y=y$$
$$d_1 (x=y) = \neg(y=y) \qquad d_2 (x=y) = \neg(y=y)$$
$$d_1 \neg(x=y) = y=y \qquad d_2 \neg(x=y) = y=y$$

Then d_1 and d_2 are defining schemas for $\Sigma(x)$. However d_1 and d_2 are clearly not equivalent. Also if $B \subseteq \overline{M}$ with $A \subset B$ then $d_1(B)$ is not a type in $S_1(B)$ since it is not complete (neither $R(x,a)$ nor $\neg R(x,a)$ are in $d_1(B)$.). Worse than that, $d_2(B)$ is not consistent. (since if $b \in B \setminus A$ then $\models \neg R(b,a)$ ie $\models d_2 R[b]$ so $R(x,b) \in d_2(B)$. Also $\models (b=b)$ ie $\models d_2 \neg R[b]$ so $\neg R(x,b) \in d_2(B)$.) Thus the condition $A \preceq \overline{M}$ cannot be eliminated in Prop 1.2.1 and Thm 1.2.2.

3. Heirs and Co-heirs:

<u>Definition:</u> Let $M \subseteq A \subseteq \overline{M}$ with $M \preceq \overline{M}$. Let $\Sigma(\underline{x}) \in S(A)$

and let $\varphi(\underline{x}, \underline{y}) \in L(M)$. We say that $\varphi(\underline{x}, \underline{y})$ is <u>represented in $\Sigma(\underline{x})$</u> iff there is a tuple $\underline{a}$ in A such that $\varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})$.

Let $\Sigma(\underline{x}) \in S_n(M)$ and $\Delta(\underline{x}) \in S_n(A)$ with $\Sigma(\underline{x}) \subseteq \Delta(\underline{x})$. We say that $\Delta(\underline{x})$ is an <u>heir of $\Sigma(\underline{x})$</u> iff for any formula $\varphi(\underline{x}, \underline{y}) \in L(M)$, if $\varphi(\underline{x}, \underline{y})$ is represented in $\Delta(\underline{x})$ then $\varphi(\underline{x}, \underline{y})$ is also represented in $\Sigma(\underline{x})$. We say that $\Delta(\underline{x})$ is a <u>co-heir of $\Sigma(\underline{x})$</u> iff for every formula $\varphi(\underline{x}) \in \Delta(\underline{x})$ there is a tuple $\underline{a}$ in M such that $\models \varphi[\underline{a}]$.

If $B \subseteq \bar{M}$ then we say that $tp(B/A)$ is an <u>heir of $tp(B/M)$</u> iff $tp(\underline{b}/A)$ is an heir of $tp(\underline{b}/M)$ for every tuple $\underline{b}$ in B. We say that $tp(B/A)$ is a <u>co-heir of $tp(B/M)$</u> iff $tp(\underline{b}/A)$ is a co-heir of $tp(\underline{b}/M)$ for every tuple $\underline{b}$ in B. (Note that when B is finite, and hence $tp(B/A)$ and $tp(B/M)$ are types, this definition is consistent with the previous one.)

For example let $M \preceq \bar{M}$, $M \subseteq A$, $\Sigma(x) \in S_1(A)$, $\Delta(x) \in S_1(M)$ with $\Delta(x) \subseteq \Sigma(x)$. If $\Sigma(x)$ is realized by some $a \in A \setminus M$ then the formula $(x = y)$ is represented in $\Sigma(x)$ (since $a = a$) but it is not represented in $\Delta(x)$ (since $a \neq b$ for all $b \in M$). Thus $\Sigma(x)$ is neither an heir nor a co-heir of $\Delta(x)$. On the other hand if $\Sigma(x)$ is realized by some $a \in M$ then clearly $\Sigma(x)$ is both an heir and a co-heir of $\Delta(x)$.

For another example consider the case where $M \preceq \bar{M}$ is the ordered structure $(\omega, <)$. Let $N \succ M$ such that there are $a, b \in N$ with $N \models (a > n) \wedge (b > n)$ for all $n \in \omega$ and $N \models (a < b)$. Suppose also that there are infinitely many elements lying between a and b. (Such an N can be found by a standard compactness

argument.) Put $\Sigma(x) = tp(b/(M \cup \{a\}))$ and $\Delta(x) = tp(b/M)$. Suppose $\varphi(x, \underline{y}) \in L(M)$ is represented in $\Sigma(x)$. Then there is a tuple $\underline{c}$ in $M \cup \{a\}$ such that $\varphi(x, \underline{c}) \in \Sigma(x)$. Then $\models \varphi[b, \underline{c}]$. Let $\underline{d}$ be a tuple in M which is identical with $\underline{c}$ except that if a is in $\underline{c}$, then a is replaced by $a' \in M$ where a' is related to the elements of $\underline{c}$ in the same way as a. (This is possible since $\underline{c}$ is finite.) Then $\models \varphi[b, \underline{d}]$ and so $\varphi(x, \underline{d}) \in \Delta(x)$. Hence $\varphi(x, \underline{y})$ is represented in $\Delta(x)$ and so we have that $\Sigma(x)$ is an heir of $\Delta(x)$. Let $\varphi(x)$ be the formula $(\exists z)((z > a) \wedge (z < x))$. Then $\varphi(x) \in L(M \cup \{a\})$ and $N \models \varphi[b]$, hence $\models \varphi[b]$. Therefore $\varphi(x) \in \Sigma(x)$ but clearly no tuple in M satisfies $\varphi(x)$. Hence $\Sigma(x)$ is not a co-heir of $\Delta(x)$.

Notice that if $\underline{b}$ is a tuple in $\overline{M}$, then $tp(\underline{b}/A)$ is an heir of $tp(\underline{b}/M)$ iff $tp(\underline{a}/(M \cup \underline{b}))$ is a co-heir of $tp(\underline{a}/M)$ for all tuples $\underline{a}$ in A, and $tp(\underline{b}/A)$ is a co-heir of $tp(\underline{b}/M)$ iff $tp(\underline{a}/(M \cup \underline{b}))$ is an heir of $tp(\underline{a}/M)$ for all tuples $\underline{a}$ in A.

We now prove a result concerning the existence of heir and co-heirs and then use this result to characterize definable types.

<u>Prop. 1.3.1</u> (Poizat [77])

Let $M \subset A \subseteq \overline{M}$ with $M \preceq \overline{M}$. If $\Sigma \in S_n(M)$ then Σ has an heir and a co-heir in $S_n(A)$.

<u>Proof:</u>

Let $\Sigma(\underline{x}) \in S_n(M)$. Let $\Delta = \{\neg\varphi(\underline{x}, \underline{a}) : \varphi(\underline{x}, \underline{y}) \in L(M)$ is not represented in $\Sigma(\underline{x})$, $\underline{a}$ a tuple in $A\} \cup \{\varphi : \varphi$ a sentence in $L(A)$ such that $\models \varphi\} \cup \Sigma(\underline{x})$. Let Γ be a finite subset of Δ. Since $\Sigma(\underline{x})$ and true sentences in $L(A)$ are closed under

conjunction we may assume that Γ is of the form:

$\{\neg\varphi_1(\underline{x},\underline{a},\underline{m}),\ldots,\neg\varphi_k(\underline{x},\underline{a},\underline{m})\}\cup\{\sigma(\underline{m},\underline{a},\underline{b})\}\cup\{\psi(\underline{m},\underline{a},\underline{b})\}$,

where $\underline{a},\underline{b}$ are tuples in $A\setminus M$, $\underline{m}$ is a tuple in M, $\sigma\in L(A)$ with $\models\sigma$ and for $i=1,\ldots,k$ $\varphi_i(\underline{x},\underline{y},\underline{m})$ is not represented in $\Sigma(\underline{x})$. Since M is a model, there are tuples $\underline{a}'$ and $\underline{b}'$ in M such that $\models\sigma(\underline{m},\underline{a}',\underline{b}')$. For $i=1,\ldots,k$ $\neg\varphi_i(\underline{x},\underline{a}',\underline{m})\in\Sigma(\underline{x})$. Thus by replacing $\underline{a}$ and $\underline{b}$ by $\underline{a}'$ and $\underline{b}'$ respectively in Γ to obtain Γ', we have $\Gamma'\subseteq\Sigma(\underline{x})$ whence Γ' and hence Γ is consistent. By compactness Δ is the consistent. So Δ has a complete and consistent extension Δ' consisting of formulae in $L(A)$ with free variables $\underline{x}$. $\{\varphi:\varphi$ a sentence in $L(A)$ such that $\models\varphi\}\subseteq\Delta'$ so $\Delta'\in S_n(A)$, and since $\Delta\subseteq\Delta'$, Δ' is an heir of $\Sigma(\underline{x})$.

Now let $\Delta=\{\varphi(\underline{x},\underline{a})\in L(A):$ for all tuples $\underline{m}$ in M, $\models\varphi(\underline{m},\underline{a})\}$. $\Delta\cup\Sigma(\underline{x})$ is consistent and so has an extension Δ' in $S_n(A)$. Every $\varphi(\underline{x})\in\Delta'$ can be satisfied by some tuple $\underline{m}$ in M and so Δ' is a co-heir of $\Sigma(\underline{x})$.

∎

Before proving the main result of this section, recall the following theorem:

<u>Prop 1.3.2</u> (Beth's Definability Theorem)

Let L be a language and put $L'=L\cup\{P\}$ where P is a new n-ary predicate symbol. Let T' be a theory in L'. If every structure M for L has at most one expansion to a structure M' for L' such that M' is a model of T', then there is a formula $\varphi(\underline{x})$ with $\underline{x}$ an n-tuple of variables such that $T'\vdash(\forall\underline{x})(P(\underline{x})\Leftrightarrow\varphi(\underline{x}))$.

∎

<u>Thm 1.3.3</u> (Poizat [77])

Let $M \preceq \bar{M}$ and let $\Sigma(\underline{x}) \in S_n(M)$. Then $\Sigma(\underline{x})$ is definable iff for all $A \supseteq M$, $\Sigma(\underline{x})$ has a unique heir in $S_n(A)$.

<u>Proof:</u>

$\Rightarrow$: Suppose d is a defining schema for $\Sigma(\underline{x})$. Let $A \supseteq M$ then by Prop 1.2.1 $d(A) \in S_n(A)$ and by Thm 1.2.2 $d(A)$ is independent of the choice of d. Suppose $\varphi(\underline{x}, \underline{y}, \underline{m}) \in L(M)$ is represented in $d(A)$ where $\underline{m}$ is a tuple in M. Then there is an $\underline{a}$ in A such that $\varphi(\underline{x}, \underline{a}, \underline{m}) \in d(A)$ and hence $\models d\varphi(\underline{a}, \underline{m})$. Thus for some $\underline{a}'$ in M, $\models d\varphi(\underline{a}', \underline{m})$ and then $\varphi(\underline{x}, \underline{a}', \underline{m}) \in \Sigma(\underline{x})$. So $d(A)$ is an heir of $\Sigma(\underline{x})$. Let $\Delta(\underline{x}) \in S_n(A)$ be an heir of $\Sigma(\underline{x})$, we have to show $\Delta(\underline{x}) = d(A)$. For all formulae $\varphi(\underline{x}, \underline{y}) \in L$ put $\psi(\underline{x}, \underline{y}) = \neg(\varphi(\underline{x}, \underline{y}) \overset{\cdot}{\Leftrightarrow} d\varphi(\underline{y}))$. Then $\psi(\underline{x}, \underline{y}) \in L(M)$ and is not represented in $\Sigma(\underline{x})$, hence not represented in $\Delta(\underline{x})$. Therefore for all $\underline{a}$ in A we have $\varphi(\underline{x}, \underline{a}) \in \Delta(\underline{x})$ iff $\models d\varphi[\underline{a}]$ iff $\varphi(\underline{x}, \underline{a}) \in d(A)$ and hence $\Delta(\underline{x}) = d(A)$.

$\Leftarrow$: For all $m \in \omega$ and $\varphi(\underline{x}, \underline{y}) \in L$ with $\underline{y}$ an m-tuple of variables add a new m-ary predicate symbol P_φ to L to obtain a language L'. Define the interpretation of P_φ in M to be $P_\varphi^M = \{\underline{a} \in M^m : \varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})\}$ and let M' be the resulting expansion of M. Let $T' = Th(M')$ in $L(M')$. Now let $(N', m)_{m \in M}$ be a model of T' and let N be the reduct of N' to L. Put $\Delta(\underline{x}) = \{\varphi(\underline{x}, \underline{a}) \in L(N) : N' \models P_\varphi[\underline{a}]\}$. Then $\Delta(\underline{x}) \in S_n(N)$, $\Delta(\underline{x})$ is an heir of $\Sigma(\underline{x})$ and $M' \prec N'$. (Recall $N \preceq M \preceq \bar{M}$ so $N \prec M$.) Since $\Delta(\underline{x})$ is a unique heir by assumption, it follows that for any structure N of L, $(N, m)_{m \in M}$ has at most one expansion to a model of T' and so by Prop. 1.3.2 for each $\varphi(\underline{x}, \underline{y}) \in L$ there is a $\psi(\underline{y}) \in L(M)$ such that $T' \vdash (\forall \underline{y})(P_\varphi(\underline{y}) \Leftrightarrow \psi(\underline{y}))$. So for all $\underline{a}$ in M, we

have $\varphi(\underline{x}, \underline{a}) \in \Sigma(\underline{x})$ iff $M \models \psi[\underline{a}]$ and hence $\Sigma(\underline{x})$ is definable

∎

If a type over a model is not definable then by the above result this type will not have unique heirs over supersets of the model. In fact we can obtain the following:

<u>Prop 1.3.4</u> (Poizat [77])
Let $M \preceq \overline{M}$ and suppose that $\Sigma(\underline{x}) \in S(M)$ is not definable. Then for all $\beta \geq |M|$, there is $N \succ M$ with $|N| \leq \beta$ such that $\Sigma(\underline{x})$ has at least β^+ heirs in $S(N)$.

(For a proof of this we refer the reader to Pillay [83] pg. 12, 13)

4. Characterization of stability by Definable Types.

We can now provide a characterization of stability using the notion of definability. First we need the following Lemma.

<u>Lemma 1.4.1</u> (Shelah [71], Pillay [83])
Let $M \preceq \overline{M}$ and let $\Sigma(\underline{x}) \in S_n(M)$ be undefinable. Then there are $N \prec M$ and $\Delta(\underline{x}) \in S_n(N)$ such that $|N| \leq \aleph_0$, $\Sigma(\underline{x})$ is an heir of $\Delta(\underline{x})$ and $\Delta(\underline{x})$ is undefinable.

<u>Proof:</u>
Let $\{M_i\}_{i<\omega}$ be a chain of countable elementary substructures of M such that for each $i<\omega$, if $\varphi(\underline{x}, \underline{y}) \in L(M_i)$ is represented in $\Sigma(\underline{x})$ then $\varphi(\underline{x}, \underline{y})$ is represented in the restriction of $\Sigma(\underline{x})$ to M_{i+1}. Put $N = \bigcup_{i<\omega} M_i$ and let $\Delta(\underline{x})$ be the restriction of $\Sigma(\underline{x})$ to N. Then $\Sigma(\underline{x})$ is an heir of $\Delta(\underline{x})$. Suppose $\Delta(\underline{x})$ is definable, then by Thm 1.3.3. $\Sigma(\underline{x})$ is the unique

heir $d(M)$ of $\Delta(\underline{x})$, where d is a defining scheme for $\Delta(\underline{x})$. Then by Prop 1.2.1 $\Sigma(\underline{x})$ is definable — contradiction. So $\Delta(\underline{x})$ is not definable.

∎

<u>Thm 1.4.2</u> (Shelah [71])

T is stable iff for all $M \preceq \overline{M}$ every type $\Sigma \in S(M)$ is definable.

<u>Proof:</u>

$\Rightarrow$: We prove this by contraposition, so assume that there is some $M \preceq \overline{M}$ and $\Sigma \in S_n(M)$ undefinable. By the above lemma we may assume that M is countable. Then by Prop 1.3.4 for all $\beta \geq \aleph_0$ there is $N \succ M$ with $|N| \leq \beta$ but $|S_n(N)| \geq \beta^+$, hence T is unstable.

$\Leftarrow$: Let β be a cardinal such that $\beta = \beta^{\aleph_0}$. Let $M \preceq \overline{M}$ with $|M| = \beta$. Let $n < \omega$. There are at most $\beta^{\aleph_0} = \beta$ defining schemes for n-types over M. By assumption all types over M are definable, so we have $|S_n(M)| \leq \beta$. Thus T is β-stable and hence stable.

∎

This theorem is a powerful tool for showing that a theory has a non-definable type or that all types are definable. This then tells us something about the uniqueness of heirs (by Thm 1.3.3). We leave examples of this to the end of the next chapter after we have developed some tools for checking if a theory is stable.

2.

Models and Theories with No order.

We now examine the notion of models and theories with 'no order' and the equivalence of stability and no order.

1. Heirs and Co-heirs of models with no order:

Definition:

Let $M \preceq \overline{M}$. We say that M has an order iff there are $n < \omega$, $\varphi(\underline{x}, \underline{y}) \in L$ with $\underline{x}, \underline{y}$ n-tuples of variables and $\underline{a}_i \in M^n$ for all $i < \omega$, such that for all $i, j < \omega$ $M \models \varphi[\underline{a}_i, \underline{a}_j]$ iff $i \leq j$. Otherwise M has no order.

We say that T has an order iff there is a model M of T such that M has an order. Otherwise T has no order.

Note that a model M might have no-order even though Th(M) has an order. For example consider the case when L has only a single binary predicate symbol $\leq$ and M is an ordered structure obtained by taking the disjoint union of arbitrary large finite chains. Let $\leq$ be interpreted by the order on M. Then M has no order since it has no infinite chains but by compactness Th(M) has a model N in which $\leq$ is interpreted by a chain order on N and N is infinite. N then has an order and hence Th(M) has an order.

Consider the model $M = (\omega, <)$ then clearly M has an order. The same is true of any infinite chain ordered structure. Let L have a single binary predicate symbol R. Let

T be the theory that says that R is an equivalence relation with n blocks for some $n<\omega$. Then no model of T has an order and so T has no order.

We will now investigate results concerning heirs and co-heirs of models with no order.

<u>Prop 2.1.1</u> (Pillay [82])

Let $M \leq \bar{M}$ and suppose M has no order. Let $\underline{a}, \underline{b}$ be tuples in $\bar{M}$ such that $tp(\underline{b}/(M \cup \underline{a}))$ is a co-heir of $tp(\underline{b}/M)$. (Equivalently $tp(\underline{a}/(M \cup \underline{b}))$ is an heir of $tp(\underline{a}/M)$.

<u>Proof:</u>

Let $tp(\underline{b}/(M \cup \underline{a}))$ be an heir of $tp(\underline{b}/M)$, but assume that it is not a co-heir of $tp(\underline{b}/M)$. Then there is a formula $\varphi(\underline{x}, \underline{y})$ in $L(M)$ such that $\models \varphi[\underline{a}, \underline{b}]$ but for all $\underline{b}'$ in M $\models \neg\varphi[\underline{a}, \underline{b}']$. We use induction to define tuples $\underline{a}_i, \underline{b}_i$ $(i<\omega)$ in M such that:

i) $\models \varphi[\underline{a}_i, \underline{b}]$ $(i<\omega)$

ii) $\models \varphi[\underline{a}_i, \underline{b}_j]$ iff $i \leq j$ $(i,j<\omega)$

Suppose that for all $i \leq n$ $(n<\omega)$ we have found tuples $\underline{a}_i$ and $\underline{b}_i$ in M satisfying conditions (i) and (ii). We have $\models \bigwedge_{i\leq n} \varphi[\underline{a}_i, \underline{b}] \wedge \bigwedge_{i \leq n} \neg\varphi[\underline{a}_i, \underline{b}_i] \wedge \varphi[\underline{a}, \underline{b}]$. Then since $tp(\underline{b}/(M \cup \underline{a}))$ is an heir of $tp(\underline{b}/M)$ there is an $\underline{a}'$ in M such that $\models \bigwedge_{i\leq n} \varphi[\underline{a}_i, \underline{b}] \wedge \bigwedge_{i\leq n} \neg[\underline{a}', \underline{b}_i] \wedge \varphi[\underline{a}', \underline{b}]$. Since $M \leq \bar{M}$ there is a $\underline{b}'$ in M such that $\models \bigwedge_{i\leq n} \varphi[\underline{a}_i, \underline{b}'] \wedge \bigwedge_{i\leq n} \neg\varphi[\underline{a}', \underline{b}_i] \wedge \varphi[\underline{a}', \underline{b}']$. Put $\underline{a}_{n+1} = \underline{a}'$, $\underline{b}_{n+1} = \underline{b}'$. Then the tuples $\underline{a}_i, \underline{b}_i$ $i \leq n+1$ satisfy (i) and (ii). We can use the same argument to find $\underline{a}_0$ and $\underline{b}_0$ and the induction is complete. By joining the constants from M which are contained in $\varphi(\underline{x}, \underline{y})$, to the tuples $\underline{b}_i$ $(i<\omega)$ and extending the tuple $\underline{y}$ of free variables, we may assume that

$\varphi(\underline{x},\underline{y}) \in L$. For all $i<\omega$ let $\underline{c}_i = (\underline{a}_i, \underline{b}_i)$ (ie the tuple obtained by joining $\underline{a}_i$ and $\underline{b}_i$). Let $\underline{v} = (\underline{x}_1, \underline{x}_2)$ and let $\underline{w} = (\underline{y}_1, \underline{y}_2)$ where $\underline{x}_1, \underline{x}_2, \underline{y}_1, \underline{y}_2$ are all tuples of variables (with the same length as $\underline{x}$ and $\underline{y}$). Put $\psi(\underline{v},\underline{w}) = \varphi(\underline{x}_1, \underline{y}_2)$. Then we have $M \models \psi[\underline{c}_i, \underline{c}_j]$ iff $M \models \varphi[\underline{a}_i, \underline{b}_j]$ iff $i \leq j$, contradicting the fact that M has no order.

∎

For our next result we need the following lemma:

<u>Lemma 2.1.2</u> (Poizat [77])

Let $M \leq \overline{M}$. Let $\Sigma \in S(M)$ and suppose that for all $A \supseteq M$ every heir of Σ in $S(A)$ is also a co-heir of Σ. Then Σ is definable.

(This lemma can be proved by considering the cardinality of the set $\{\Sigma \in S_n(A) : \Sigma$ is a co-heir of $\Sigma \cap L(M)\}$ and using the results of the previous chapter. For details we refer the reader to Pillay [83] pp 14.)

<u>Prop 2.1.3</u> (Pillay [82])

Let $M \leq \overline{M}$ and suppose M has no order. Let $\Sigma \in S(M)$ and let $A \subseteq M$. Then Σ has a unique heir Σ' in $S(A)$ which is also a unique co-heir.

<u>Proof:</u>

Let $\Sigma' \in S(A)$ be an heir of Σ. If $\varphi \in \Sigma'$ then there is some tuple $\underline{a}$ in A such that $\varphi \in \Sigma' \cap L(M \cup \underline{a})$. $\Sigma' \cap L(M \cup \underline{a})$ is an heir of Σ, therefore since M has no order we have by Prop 2.1.1 that $\Sigma' \cap L(M \cup \underline{a})$ is a co-heir of Σ. Hence φ is satisfied by some tuple in M, and so Σ' is a co-heir of Σ. By Lemma 2.1.2 we then have that Σ is definable,

and so Σ has a unique heir which by the above argument is also a co-heir. In fact the unique heir is $d(A)$ where d is a defining scheme for Σ. If Σ' is any co-heir of Σ then by a similar argument to the one above, Σ' is unique and so $\Sigma' = d(A)$. Thus $d(A)$ is a unique heir and co-heir of Σ.

∎

For our next result and for the following chapter we need to introduce the following concepts:

<u>Definition:</u>

Let $A \subseteq \overline{M}$ and let $\Sigma(\underline{x}) \in S_n(A)$. Then we define the <u>class of Σ</u> to be the set:

$$cl(\Sigma) = \{\varphi(\underline{x}, \underline{y}) \in L : \varphi(\underline{x}, \underline{y}) \text{ is represented in } \Sigma(\underline{x})\}$$

We define the <u>fundamental order for n-types of T</u> to be the ordered structure:

$$O_n(T) = (\{cl(\Sigma) : \Sigma \in S_n(M),\ M \preceq \overline{M}\},\ \subseteq)$$

If $\Sigma(\underline{x}) \in S_n(M)$ and $\Delta(\underline{x}) \in S_n(N)$ where $M, N \preceq \overline{M}$ we write $\Sigma \leq \Delta$ if $cl(\Sigma) \subseteq cl(\Delta)$ and we write $\Sigma \sim \Delta$ if $cl(\Sigma) = cl(\Delta)$.

<u>Prop 2.1.4</u> (C. Berline in Pillay [83])

Let T have no order. Let $M \preceq A \preceq \overline{M}$ with $M \preceq \overline{M}$ and let $\Sigma(\underline{x}) \in S_n(M)$, $\Delta(\underline{x}) \in S_n(A)$ with $\Sigma \subseteq \Delta$. Then $\Sigma \sim \Delta$ implies that Δ is a unique heir and co-heir of Σ in $S_n(A)$.

Proof:

Suppose $\Sigma \sim \Delta$ but that Δ is not a co-heir of Σ. Let $\underline{b}$ be a tuple in $\bar{M}$ realizing Δ. Then there is some $\varphi(\underline{y}, \underline{z}) \in L$ and $\underline{a}$ in M such that $\models \varphi[\underline{a}, \underline{b}]$ but for all $\underline{m}$ in M, $\not\models \varphi[\underline{a}, \underline{m}]$. We use induction to define tuples $\underline{a}_{i,k}$ and $\underline{b}_{j,k}$ in M for $k < \omega$ and $i \leq k$ such that the following conditions hold:

(i) $M \models \varphi[\underline{a}_{i,k}, \underline{b}_{j,k}]$ iff $i \leq j$ $(i, j \leq k)$

(ii) $\models \varphi[\underline{a}_{i,k}, \underline{b}]$ $(i \leq k)$

Suppose we have found $\underline{a}_{i,m}$ and $\underline{b}_{i,m}$ in M for $i \leq m$, satisfying (i) and (ii) for $k = m$. Let $\varphi^{i \leq j}$ denote φ if $i \leq j$ and $\neg\varphi$ if $i > j$. Then $\models \bigwedge_{i,j \leq m} \varphi^{i \leq j}[\underline{a}_{i,m}, \underline{b}_{j,m}] \wedge \bigwedge_{i \leq m} \varphi[\underline{a}_{i,m}, \underline{b}] \wedge \bigwedge_{i \leq m} \neg\varphi[\underline{a}, \underline{b}_{i,m}] \wedge \varphi[\underline{a}, \underline{b}]$. We have $cl(\Sigma) = cl(\Delta)$ so we can replace $\underline{a}_{i,m}$ by $\underline{a}'_i$, $\underline{b}_{i,m}$ by $\underline{b}'_i$ (for $i \leq m$), and $\underline{a}$ by $\underline{a}'$, in the expression above, where $\underline{a}'_i$, $\underline{b}'_i$ $(i \leq m)$ and $\underline{a}'$ are tuples in M. We may also replace $\underline{b}$ by some tuple $\underline{b}'$ in M, since $M \preceq \bar{M}$. For $i \leq m$ put $\underline{a}_{i,m+1} = \underline{a}'_i$, $\underline{b}_{i,m+1} = \underline{b}'_i$ and put $\underline{a}_{m+1,m+1} = \underline{a}'$ and $\underline{b}_{m+1,m+1} = \underline{b}'$. Then (i) and (ii) are satisfied for $k = m+1$. We may use the same argument to find $\underline{a}_{0,0}$ and $\underline{b}_{0,0}$ and the induction is complete. By compactness we then have tuples $\underline{a}_i$ and $\underline{b}_i$ for $i < \omega$ in some $N \preceq \bar{M}$ such that $\models \varphi[\underline{a}_i, \underline{b}_j]$ iff $i \leq j$. But then as in the proof of Prop 2.1.1, N has an order, thus contradicting the fact that T has no order. Thus Δ is a co-heir of Σ and since T has no order, M has no order and so by Prop 2.1.3. Δ must be the unique heir and co-heir of Σ in $Sn(A)$.

∎

2. <u>Equivalence of stability and No order</u>

We wish to show that a theory is stable iff it has no order.

In order to obtain this result we need to introduce the concept of the 'rank' of a formula.

<u>Definition:</u>

Let $\varphi(\underline{x}, \underline{y}) \in L$ where $\underline{x}$ is an n-tuple of variables. Let $\psi(\underline{x}) \in L(A)$ $(A \subseteq \bar{M})$. We define the ordinal $R_n^{\varphi}(\psi)$ inductively as follows:

(i) $R_n^{\varphi}(\psi) \geq 0$ if ψ is consistent with T.

(ii) If α is a limit ordinal then $R_n^{\varphi}(\psi) \geq \alpha$ iff $R_n^{\varphi}(\psi) \geq \beta$ for all $\beta < \alpha$.

(iii) $R_n^{\varphi}(\psi) \geq \alpha + 1$ iff there is a tuple $\underline{b}$ such that $R_n^{\varphi}(\psi \wedge \varphi(\underline{x}, \underline{b})) \geq \alpha$ and $R_n^{\varphi}(\psi \wedge \neg\varphi(\underline{x}, \underline{b})) \geq \alpha$

Then we put $R_n^{\varphi}(\psi) = \alpha$ if $R_n^{\varphi}(\psi) \geq \alpha$ and $R_n^{\varphi}(\psi) \not\geq \alpha + 1$.
If $R_n^{\varphi}(\psi) \geq \alpha$ for all ordinals α we put $R_n^{\varphi}(\psi) = \infty$ where we consider $\alpha < \infty$ for all α.

If $\Sigma(\underline{x}) \in S_n(A)$ $(A \subseteq \bar{M})$ we define $R_n^{\varphi}(\Sigma)$ to be:

$$R_n^{\varphi}(\Sigma) = \min\{R_n^{\varphi}(\psi) : \psi \in \Sigma\}.$$

The ordinals $R_n^{\varphi}(\psi)$ and $R_n^{\varphi}(\Sigma)$ are referred to as ranks.
(The definition above is based on Shelah's rank $R^n(\Sigma, \varphi, \alpha)$ — see Shelah [78]. It should not be confused with Morley's rank.)
Note that by (i) $R_n^{\varphi}(\psi)$ depends on T. We will abbreviate R_n^{φ} by R^{φ} if the value of n is clear.
We have the following immediate results:

<u>Prop. 2.2.1</u>

i) Let $\psi_1(\underline{x}), \psi_2(\underline{x}) \in L(A)$ and let $\varphi(\underline{x}, \underline{y}) \in L$. Suppose that $\models (\psi_1(\underline{x}) \Rightarrow \psi_2(\underline{x}))$. Then $R^{\varphi}(\psi_1) \leq R^{\varphi}(\psi_2)$.

ii) Let $\Sigma(\underline{x}) \in S_n(A)$, $\Delta(\underline{x}) \in S_n(B)$ and let $\Sigma \subseteq \Delta$, where $A \subseteq B \subseteq \overline{M}$. Then $R^{\varphi}(\Sigma) \geq R^{\varphi}(\Delta)$ for any $\varphi(\underline{x}, \underline{y}) \in L$.

iii) Let $\Sigma(\underline{x}) \in S_n(A)$ and let $\varphi(\underline{x}, \underline{y}) \in L$, then there is a $\psi(\underline{x}) \in \Sigma(\underline{x})$ such that $R^{\varphi}(\Sigma) = R^{\varphi}(\psi)$.

iv) Let $\psi(\underline{x}, \underline{z})$, $\varphi(\underline{x}, \underline{y}) \in L$ and $n < \omega$. Then "$R^{\varphi}(\psi) \geq n$" is an elementary property of $\underline{z}$ (ie there is some $\sigma(\underline{z}) \in L$ such that $R^{\varphi}(\psi) \geq n$ iff $\models (\forall \underline{z}) \sigma(\underline{z})$

For an example of $R^{\varphi}(\psi)$ consider $\psi(\underline{x}) \in L$. Define $Sat(\psi)$ to be:

$$Sat(\psi) = \{ \underline{b} \in \overline{M} : \models \psi[\underline{b}] \}$$

ie $Sat(\psi)$ is the set of tuples satisfying $\psi(\underline{x})$.
Consider the case where $\psi(x)$ is a consistent formula in one free variable x. Let $\varphi(x, y)$ be the formula $(x = y)$.
Computing $R^{\varphi}(\psi)$ from the definition we have

$$R^{\varphi}(\psi) = \begin{cases} 0 & \text{if } |Sat(\psi)| = 1 \\ 1 & \text{if } |Sat(\psi)| > 1 \end{cases}$$

We will only be concerned with using R^{φ} as an aid for proving Thm 2.2.3. For more information on R^{φ} and techniques of computation we refer the reader to Pillay [83] pg 21, pg 68 – 86.

Before proving the main result of this section we need the following lemma:

<u>Lemma 2.2.2</u> (Shelah [78])

Let $\varphi(\underline{x}, \underline{y}) \in L$ and let $\underline{a}_i$ for $i < 2^n$ $(n < \omega)$ be such that $\models \varphi[\underline{a}_i, \underline{a}_j]$ iff $i \leq j$ (for $i, j < 2^n$). Let $\psi(\underline{x}) \in L(A)$ $(A \subseteq \overline{M})$ and suppose $\models \psi[\underline{a}_i]$ for all $i < 2^n$. Then $R^{\varphi}(\psi) \geq n$

Proof:

We prove the result by induction. For $n=0$ the result is trivial. Assume the result holds for $n \leq k$ and let φ, ψ and $\underline{a}_i$ for $i < 2^n$ satisfy the hypotheses of the lemma for $n=k+1$. Then for $0 \leq i < 2^k$, $\underline{a}_i$ satisfies $\psi(\underline{x}) \wedge \neg\varphi(\underline{a}_{2^k}, \underline{x})$, and for $2^k \leq i \leq 2^{k+1}$, $\underline{a}_i$ satisfies $\psi(\underline{x}) \wedge \varphi(\underline{a}_{2^k}, \underline{x})$. Thus by the inductive hypothesis we have $R^\varphi(\psi(\underline{x}) \wedge \varphi(\underline{a}_{2^k}, \underline{x})) \geq k$ and $R^\varphi(\psi(\underline{x}) \wedge \neg\varphi(\underline{a}_{2^k}, \underline{x})) \geq k$. Thus by the definition of $R^\varphi(\psi)$ we have $R^\varphi(\psi) \geq k+1$ and the induction is complete.

■

Thm: 2.2.3 (Shelah [78])

The following are equivalent:

(a) T is stable.

(b) For all $\varphi(\underline{x}, \underline{y}) \in L$ $R^\varphi(\underline{x}=\underline{x}) < \omega$.

(c) T has no order.

Proof:

(a) $\Rightarrow$ (b):

Let T be stable but suppose there is some $\varphi(\underline{y}, \underline{x}) \in L$ with $R^\varphi(\underline{x}=\underline{x}) \geq \omega$. Let $\Gamma(\alpha)$ denote the following set of formulae for α an ordinal:

$$\{\underline{x}_f = \underline{x}_f : f \in {}^\alpha 2\} \cup \{\varphi(\underline{x}_{f|\beta}) : f \in {}^\alpha 2, \beta < \alpha, f(\beta) = 0\}$$
$$\cup \{\neg\varphi(\underline{x}_{f|\beta}) : f \in {}^\alpha 2, \beta < \alpha, f(\beta) = 1\}$$

By induction we can show that $\Gamma(n)$ is consistent iff $R^\varphi(\underline{x}=\underline{x}) \geq n$ $(n<\omega)$. Since $R^\varphi(\underline{x}=\underline{x}) \geq \omega$ we thus have $\Gamma(n)$ is consistent for all $n<\omega$. By relabelling variables we have that all finite subsets of $\Gamma(\alpha)$ are consistent, and hence by compactness $\Gamma(\alpha)$ is consistent, for any ordinal. Let λ be an infinite cardinal and let μ be the least cardinal such that $2^\mu > \lambda$. Since $\Gamma(\mu)$ is consistent there are tuples $\underline{b}_f$ for $f \in {}^\mu 2$

and $\underline{a}_h$ for $h \in {}^{\mu>}2$ such that for $f \in {}^{\mu}2$ and $\beta < \mu$ we have $\models \varphi[\underline{b}_f, \underline{a}_{f|\beta}]$ if $f(\beta) = 0$ and $\models \neg\varphi[\underline{b}_f, \underline{a}_{f|\beta}]$ if $f(\beta) = 1$. Let A be the set of all the constants in the tuples $\underline{a}_h$ ($h \in {}^{\mu>}2$). Then $|A| \leq \lambda$. If $f, g \in {}^{\mu}2$ and $f \neq g$, then $tp(\underline{b}_f / A) \neq tp(\underline{b}_g / A)$ and so $|S(A)| > |A|$. Hence T is not λ-stable for any λ, which contradicts the fact that T is stable.

(b) $\Rightarrow$ (c.)

Assume (b.) but assume that T has an order. Then there is some $M \preceq \bar{M}$ with tuples $\underline{a}_i$ for $i < \omega$, in M and some $\varphi(\underline{y}\ \underline{x})$ such that $\models \varphi[\underline{a}_i, \underline{a}_j]$ iff $i \leq j$. By Lemma 2.2.2 we see that for all $n < \omega$ $R^{\varphi}(\underline{x} = \underline{x}) \geq n$ contradicting $R^{\varphi}(\underline{x} = \underline{x}) < \omega$.

(c.) $\Rightarrow$ (a.) (for alternative approach see Hodges [81])

Let T have no order. Then for all $M \preceq \bar{M}$, M has no order. Hence by Prop 2.1.3. for all $M \preceq \bar{M}$ each type in $S(M)$ has a unique heir and is therefore definable by Thm 1.3.3. Then by Thm 1.4.2 T is stable.

■

We already know that if T is stable and $M \preceq \bar{M}$, then every type in $S(M)$ is definable. We can now generalize this result to any $M \subseteq \bar{M}$ where M need not be a submodel.

<u>Corollary 2.2.4</u> (Shelah [78])

Let T be stable and let $M \subseteq \bar{M}$. Then every type in $S(M)$ is definable.

<u>Proof:</u>

Let $\varphi(\underline{x}, \underline{y}) \in L$. By Thm 2.2.3 and Prop 2.2.1 (i) we have

that for all $\psi(\underline{x}) \in L(A)$ $(A \subseteq \overline{M})$ $R^{\varphi}(\psi) < \omega$. Therefore if $\Sigma(\underline{x}) \in S(M)$ by Prop 2.2.1 (iii) there is some $\psi(\underline{x}) \in \Sigma(\underline{x})$ such that $R^{\varphi}(\Sigma) = R^{\varphi}(\psi) = n_{\varphi}$ say, where $n_{\varphi} < \omega$. Then for any tuple $\underline{b}$, either $R^{\varphi}(\psi(\underline{x}) \wedge \varphi(\underline{x}, \underline{b})) < n_{\varphi}$ or $R^{\varphi}(\psi(\underline{x}) \wedge \neg\varphi(\underline{x}, \underline{b})) < n_{\varphi}$. Hence for any $\underline{m}$ in M we have $\varphi(\underline{x}, \underline{m}) \in \Sigma(\underline{x})$ iff $R^{\varphi}(\psi(\underline{x}) \wedge \varphi(\underline{x}, \underline{m})) \geq n_{\varphi}$. Hence by Prop 2.2.1 (iv) $\varphi(\underline{x}, \underline{m}) \in \Sigma(\underline{x})$ iff $\models \sigma[\underline{m}]$ for some $\sigma(\underline{z}) \in L$, and so $\Sigma(\underline{x})$ is definable.

$\blacksquare$

We end this chapter by summarizing the various equivalences of stability which we now know and giving examples of their application.

<u>Prop 2.2.5</u>

The following are equivalent:

(a) T is stable

(b.) For all $M \leq \overline{M}$ every $\Sigma \in S(M)$ is definable

(c.) For all $M \leq \overline{M}$ every $\Sigma \in S(M)$ has a unique heir in $S(A)$ for all $A \supseteq M$.

(d.) T has no order.

(e.) For all $\varphi(\underline{x}, \underline{y}) \in L$, $R^{\varphi}(\underline{x} = \underline{x}) < \omega$

$\blacksquare$

If M is the disjoint union of arbitrary large finite chains, then as we have mentioned on pg 18. $Th(M)$ has an order and so by the above $Th(M)$ cannot be stable. We also have immediately that there is some model N of $Th(M)$ which has an undefinable type. This shows that not all types are definable. Also there is a model N of $Th(M)$ which has a type that has more than one heir over some superset of N. This shows the power of stability theory as a tool for showing the existence of certain types which may be required in a proof of some result.

Let T be the theory of an infinite set. Then clearly T is ω-stable and hence stable. Then we have by corollary 2.2.4 that all types over subsets of models of T are definable.

Consider $T = Th(\mathbb{Q}, \leq)$. Then T has an order. Hence we can find a formula $\varphi(\underline{x}, \underline{y}) \in L$ such that $R^{\varphi}(\underline{x} = \underline{x}) \geq \omega$. Let T be the theory of infinite abelian groups with all elements of order p for p prime. Then for any $\varphi(\underline{x}, \underline{y}) \in L$ we have $R^{\varphi}(\underline{x} = \underline{x}) < \omega$ (by induction on the complexity of formulae). Hence T is stable and T has no order.

3.

Forking

In chapter 1 we introduced the notions of heirs and co-heirs of types. These were special extensions of a type. In this chapter we examine more general extensions of types in the case of stable theories. Most of the results are originally due to Shelah [78]. (These are the results below with no references.)

1. Stability and Forking:

Definition:

Let $A \subseteq \bar{M}$ and let $\Sigma \in S_n(A)$. We define C_Σ by:

$$C_\Sigma = \{ d(\Delta) : \Delta \in S_n(M), \Sigma \subseteq \Delta, \text{ where } A \subseteq M \preceq \bar{M} \}$$

Notice that C_Σ is a subset of $On(T)$, and is also ordered by $\subseteq$.

Prop 3.1.1

Let $A \subseteq \bar{M}$ and $\Sigma(\underline{x}) \in S(A)$. Then C_Σ has a minimal element.

Proof:

We use the dual of Zorn's Lemma. Let $\mathcal{C}$ be a chain in C_Σ. Let $M \preceq \bar{M}$ such that $A \subseteq M$. Put $\Gamma(\underline{x}) = \Sigma(\underline{x}) \cup \{ \neg\varphi(\underline{x}, \underline{c}) \in L : \underline{c}$ is a tuple in M, $\varphi(\underline{x}, \underline{y}) \notin \theta$ for some $\theta \in \mathcal{C} \}$. Let $\mathcal{C}_0$ be a finite subset of $\mathcal{C}$. Let $\Gamma_0(\underline{x}) = \Sigma(\underline{x}) \cup \{ \neg\varphi(\underline{x}, \underline{c}) \in \Gamma(\underline{x}) : \varphi(\underline{x}, \underline{y}) \notin \theta$ for some $\theta \in \mathcal{C}_0 \}$. We may assume that $\mathcal{C}_0$ has a lower bound $\theta_0 \in \mathcal{C}$. Let $N \preceq \bar{M}$ with $A \subseteq N$ and let $N \preceq \bar{M}$ with $A \subseteq N$ and let $\Delta(\underline{x}) \in S(N)$ with $\Sigma(\underline{x}) \subseteq \Delta(\underline{x})$ and $d(\Delta) = \theta_0$. Let $\Delta'(\underline{x})$ be an heir of $\Delta(\underline{x})$ over $M \cup N$. Then we have

$cl(\Delta') = \theta_0$. Then $cl(\Delta' \cap L(M)) \subseteq \theta_0$ and hence $\Gamma_0(\underline{x}) \subseteq \Delta'(\underline{x}) \cap L(M)$. Thus $\Gamma_0(\underline{x})$ is consistent and so by compactness $\Gamma(\underline{x})$ is consistent. Extend $\Gamma(\underline{x})$ to a complete type $\Gamma'(\underline{x}) \in S(M)$. Then for all $\theta \in \mathcal{C}$, $cl(\Gamma') \subseteq \theta$ and so $cl(\Gamma')$ is a lower bound for $\mathcal{C}$. Clearly C_Σ is not empty and so by the dual of Zorn's Lemma we have that C_Σ has a minimal element. ∎

If $A \subseteq M \preceq \bar{M}$, $\Sigma \in S_n(A)$, $\Delta \in S_n(M)$ with $\Sigma \subseteq \Delta$ and Δ is realized by a tuple that is in fact in M, then $cl(\Delta)$ is a maximal element of C_Σ. However we are not concerned with maximal elements.

We can generalize the notion of types to include sets of formulae in free variables from an arbitrary set of variables $\{x_i : i \in I\}$. $tp(B/A)$ where B is infinite is an example of such a generalized type. All the definitions concerning types, such as heirs, co-heirs, $On(T)$, class and C_Σ can then be generalized for types of this kind. Clearly Prop 3.1.1 will still hold for generalized types.

The minimal element of C_Σ whose existence is given by Prop. 3.1.1 need not be unique. However, when T is stable we get the following result.

<u>Prop 3.1.2</u> (Proof following Lascar and Poizat [79])
Let T be stable. Let $A \subseteq \bar{M}$ and let $\Sigma \in S(A)$. Then C_Σ has a unique minimal element (which is thus the minimum element of C_Σ.)

<u>Proof:</u>
Let $\underline{c}$ be a tuple in $\bar{M}$ realizing Σ. Let $M_1 \preceq \bar{M}$ with $A \cup \underline{c} \subseteq M_1$. Then by the generalization of Prop 3.1.1 we

have that there is $M \leq \overline{M}$ with $A \subseteq M$ such that $cl(tp(M/M_1))$ is a minimal element of $C_{tp(M/A)}$. Put $T = cl(tp(\underline{c}/M))$. We wish to show that T is a minimum element of C_Σ. Let $M' \leq \overline{M}$ such that $A \subseteq M'$ and let $\Sigma' \in S(M)$ be an extension of Σ. If necessary we can replace M' by another model isomorphic to M' over A, so that we have $\Sigma' = tp(\underline{c}/M')$ and $tp(M'/(M \cup M_1))$ is the co-heir of $tp(M'/M_1)$. Then we have $tp(M/(M' \cup M_1))$ is the heir of $tp(M/M_1)$. Therefore $tp(M/M') \leq tp(M/(M_1 \cup M'))$ and $tp(M/(M_1 \cup M')) \sim tp(M/M_1)$. By the choice of M we have that $tp(M/M_1) \leq tp(M/M')$ and so we get $tp(M/M') \sim tp(M/(M_1 \cup M'))$. Now T is stable and hence by the equivalence of stability and no order and prop. 2.1.4. we have that $tp(M/(M_1 \cup M'))$ is a co-heir of $tp(M/M')$. Hence $tp(M_1/(M \cup M'))$ is the heir of $tp(M_1/M')$, and so we have $tp(M_1/M') \sim tp(M_1/(M \cup M')) \geq tp(M_1/M)$. $\underline{c}$ is a tuple in M_1 and so $tp(\underline{c}/M') \geq tp(\underline{c}/M)$ ie $\Sigma' \geq tp(\underline{c}/M)$. Hence T is the minimum of C_Σ.

■

To show that the condition of T being stable cannot be eliminated in the above result, consider $T = Th(\mathbb{Q}, <)$. Let $\Sigma(x)$ be the unique 1-type over $\emptyset$. ($\Sigma(x)$ is axiomatized by $(x = x)$). Let M be a model of T. Let $\Delta_1(x) \in S_1(M)$ be the type which contains $(x < a)$ for all $a \in M$ and let $\Delta_2(x) \in S_1(M)$ be the type which contains $(x > a)$ for all $a \in M$. Then $cl(\Delta_1)$ and $cl(\Delta_2)$ are both minimal in C_Σ.

Prop. 3.1.2 allows us to make the following definition.

<u>Definition:</u>

Let T be stable and let $A \subseteq \overline{M}$. Let $\Sigma \in S(A)$. We call the

minimum element of C_Σ, the <u>bound</u> of Σ.

We denote the bound of Σ by $\beta(\Sigma)$.

Note that if $M \preceq \overline{M}$ such that $A \subseteq M$ then we can find an extension Σ' of Σ in $S(M)$ such that $cl(\Sigma') = \beta(\Sigma)$.
(Let $N \preceq \overline{M}$ such that $A \subseteq N$ and let Δ be an extension of Σ to $S(N)$ such that $cl(\Delta) = \beta(\Sigma)$. Let Δ' be the heir of Δ over $M \cup N$. The restriction of Δ' to M gives the required Σ'.)

<u>Definition</u>:
Let T be stable. Let $A \subseteq B \subseteq \overline{M}$ and let $\Sigma \in S_n(A)$, $\Delta \in S_n(B)$ with $\Sigma \subseteq \Delta$. We say that Δ <u>forks over A</u> iff $\beta(\Delta) \neq \beta(\Sigma)$.
If Δ forks over A we say that Δ is a <u>forking extension</u> of Σ, otherwise Δ is a <u>non-forking extension</u>

Note that Δ is a non-forking extension of Σ iff $\beta(\Delta) = \beta(\Sigma)$.

If $A \subseteq B \subseteq \overline{M}$ and $\Sigma \in S(B)$ we denote the restriction of Σ to A by $\Sigma|_A$. (ie $\Sigma|_A = \Sigma \cap L(A)$.)

<u>Thm 3.1.3</u>
Let T be stable. Then:
a) If $M \subseteq A \subseteq \overline{M}$, $M \preceq \overline{M}$ and $\Sigma \in S(A)$ then Σ does not fork over A iff Σ is the heir of $\Sigma|_M$ (iff Σ is definable over M).

b) If $A \subseteq \overline{M}$ and $\Sigma \in S(A)$, then Σ does not fork over A.

c) Let $A \subseteq B \subseteq C \subseteq \overline{M}$ and let $\Sigma \in S(C)$. Then Σ does not fork over A iff Σ does not fork over B and $\Sigma|_B$ does

not fork over A.

d.) Let $A \subseteq B \subseteq \bar{M}$ and let $\Sigma \in S_n(A)$. Then Σ has a non-forking extension Δ in $S_n(B)$.

Proof:

a) Since M is a model we have $\beta(\Sigma|_M) = cl(\Sigma|_M)$ and $\beta(\Sigma) = cl(\Sigma)$. Hence we have that Σ does not fork over M iff $\beta(\Sigma|_M) = \beta(\Sigma)$ iff $cl(\Sigma|_M) = cl(\Sigma)$ iff Σ is an heir of $\Sigma|_M$. (By Prop 2.1.4.).

b) Follows from the definition of forking.

c.) Suppose that Σ does not fork over A. Then we have $\beta(\Sigma|_A) = \beta(\Sigma)$. Now $\beta(\Sigma|_A) \subseteq \beta(\Sigma|_B) \subseteq \beta(\Sigma)$ and so $\beta(\Sigma|_B) = \beta(\Sigma)$ ie Σ does not fork over B. Let $\Delta = \Sigma|_B$. Then $\beta(\Delta|_A) = \beta(\Sigma|_A) = \beta(\Sigma) = \beta(\Sigma|_B) = \beta(\Delta)$ ie $\Sigma|_B$ does not fork over A.

Suppose that Σ does not fork over B and $\Sigma|_B$ does not fork over A. Then if $\Delta = \Sigma|_B$ we have:

$\beta(\Sigma) = \beta(\Sigma|_B) = \beta(\Delta) = \beta(\Delta|_A) = \beta(\Sigma|_A)$, and so Σ does not fork over A.

d.) Let $M \preceq \bar{M}$ with $B \subseteq M$. Then there is $\Gamma \in S_n(M)$ such that $cl(\Gamma) = \beta(\Sigma)$. But since M is a model $\beta(\Gamma) = cl(\Gamma)$ and so $\beta(\Gamma) = \beta(\Sigma)$ ie Γ does not fork over A. Hence by (c.) $\Gamma|_B$ does not fork over A and hence $\Delta = \Gamma|_B$ is a non-forking extension of Σ in $S_n(B)$.

The next result is an extension of prop 2.1.1.

<u>Prop. 3.1.4</u> (Proof following Lascar and Poizat [79])

Let T be stable and $A \subseteq \overline{M}$. Let $\underline{a}$ and $\underline{b}$ be tuples in $\overline{M}$. Then $tp(\underline{a}/(A \cup \underline{b}))$ forks over A iff $tp(\underline{b}/(A \cup \underline{a}))$ forks over A. (Forking symmetry.)

<u>Proof:</u>

Suppose $tp(\underline{b}/(A \cup \underline{a}))$ does not fork over A. Let $M \preceq \overline{M}$ with $A \cup \underline{a} \subseteq M$. By Thm 3.1.3 there is a non-forking extension $\Sigma \in S(M)$ of $tp(\underline{b}/(A \cup \underline{a}))$. We may assume that $\underline{b}$ realizes Σ and so $tp(\underline{b}/M)$ does not fork over A. We can choose a model $N \preceq \overline{M}$ such that:

(i) $A \subseteq N$

(ii) $tp(\underline{a}/N)$ does not fork over A, and

(iii) $tp(\underline{b}/(M \cup N))$ is the heir of $tp(\underline{b}/M)$.

Then by (iii), Thm 3.1.3(c) and the fact that $tp(\underline{b}/M)$ does not fork over A we have that $tp(\underline{b}/(M \cup N))$ does not fork over A. Hence by Thm 3.1.3 (c) $tp(\underline{b}/(N \cup \underline{a}))$ does not fork over N and is thus the heir of $tp(\underline{b}/N)$. Hence $tp(\underline{a}/(N \cup \underline{b}))$ is the heir of $tp(\underline{a}/N)$. Now $tp(\underline{a}/N)$ does not fork over A and so $tp(\underline{a}/(N \cup \underline{b}))$ does not fork over A. By Thm 3.1.3 (c) we then have that $tp(\underline{a}/(A \cup \underline{b}))$ does not fork over A. Thus if $tp(\underline{a}/(A \cup \underline{b}))$ forks over A we have $tp(\underline{b}/(A \cup \underline{a}))$ forks over A and similarly the converse holds. ∎

<u>Prop 3.1.5</u>

Let T be stable. Let $A \subseteq B \subseteq \overline{M}$ and let $\Sigma(\underline{x}) \in S(A)$ and $\Delta(\underline{x}) \in S(B)$ with $\Sigma(\underline{x}) \subseteq \Delta(\underline{x})$. Then $\Delta(\underline{x})$ does not fork over A iff for any $M \preceq \overline{M}$ with $B \subseteq M$ the set of formulae $\Phi_M(\underline{x}) = \Delta(\underline{x}) \cup \{\neg\varphi(\underline{x}, \underline{a}) : \underline{a}$ a tuple in M, $\varphi(\underline{x}, \underline{y})$ in L and $\varphi(\underline{x}, \underline{y}) \notin \beta(\Sigma)\}$ is consistent with $\overline{M}$.

Proof:

$\Rightarrow$: Let $M \preceq \bar{M}$ with $B \subseteq M$. Since $\Delta(\underline{x})$ does not fork over A, $\Delta(\underline{x})$ has an extension $\Delta'(\underline{x}) \in S(M)$ which does not fork over A, ie $\beta(\Delta') = \beta(\Sigma)$. But M is a model, so we have $cl(\Delta') = \beta(\Delta') = \beta(\Sigma)$. Hence $\Phi_M(\underline{x}) \subseteq \Delta'(\underline{x})$ and so $\Phi_M(\underline{x})$ is consistent with $\bar{M}$.

$\Leftarrow$: Let $M \preceq \bar{M}$ with $B \subseteq M$. Since $\Phi_M(\underline{x})$ is consistent, there is a $\Delta'(\underline{x}) \in S(M)$ extending $\Phi_M(\underline{x})$. Then we have that $\beta(\Sigma) = cl(\Delta') = \beta(\Delta')$ and so $\Delta'(\underline{x})$ does not fork over A. Hence $\Delta(\underline{x})$ does not fork over A.

■

Prop 3.1.6

Suppose T is stable, $A \subseteq B \subseteq \bar{M}$, and $\Sigma \in S(B)$ forks over A. Then there is a tuple $\underline{b}$ in B such that $\Sigma|_{A \cup \underline{b}}$ forks over A.

Proof:

By Prop 3.1.5 there is a $M \preceq \bar{M}$ with $B \subseteq M$ such that the set $\Sigma(\underline{x}) \cup \{\neg\varphi(\underline{x}, \underline{a}) : \underline{a}$ a tuple in M, $\varphi(\underline{x}, \underline{y})$ in L and $\varphi(\underline{x}, \underline{y}) \notin \beta(\Sigma|_A)\}$ is not consistent with. There is some finite subset Σ_0 of Σ which is responsible for the inconsistency. Let $\underline{b}$ be the tuple of constants from B occurring in formulae from Σ_0. Then $\Sigma|_{A \cup \underline{b}} \cup \{\neg\varphi(\underline{x}, \underline{a}) : \underline{a}$ a tuple in M, $\varphi(\underline{x}, \underline{y})$ in L and $\varphi(\underline{x}, \underline{y}) \notin \beta(\Sigma|_A)\}$ is also not consistent with $\bar{M}$ and so by Prop 3.1.5 we have that $\Sigma|_{A \cup \underline{b}}$ does not fork over A.

■

We have the following corollary:

<u>Corollary 3.1.7</u>

Let T be stable. Let $\{A_\beta : \beta < \alpha\}$ be a chain of subsets of $\overline{M}$, such that for all $\beta < \alpha$ $tp(\underline{a}/A_\beta)$ does not fork over A_0, where $\underline{a}$ is a tuple in $\overline{M}$. Then $tp(\underline{a}/\bigcup_{\beta<\alpha} A_\beta)$ does not fork over A_0.

(To prove this simply assume that $tp(\underline{a}/\bigcup_{\beta<\alpha} A_\beta)$ does fork over A_0 and apply Prop 3.1.6 to obtain a contradiction.)

2. Independence

We now extend the notion of forking to generalized types and define the notion of 'independence'.

<u>Definition</u>

Let T be stable and let $A, B, C \subseteq \overline{M}$. We say that $tp(B/(A \cup C))$ <u>forks over A</u> iff there is a tuple $\underline{b}$ in B such that $tp(\underline{b}/(A \cup C))$ forks over A. We say that B, C are <u>independent over A</u> iff $tp(B/(A \cup C))$ does not fork over A.

Note that by forking symmetry (Prop 3.1.4) we have that $tp(B/(A \cup C))$ forks over A iff $tp(C/(A \cup B))$ forks over A. Hence 'independence over A' is a well-defined symmetric relation.

We also define independence for families of sets. Let $\{B_i : i \in I\}$ be a family of subsets of $\overline{M}$. We say that $\{B_i : i \in I\}$ is <u>independent over A</u> iff for all $i \in I$ $tp(B_i/(\bigcup\{B_j : j \in I \setminus \{i\}\} \cup A))$ does not fork over A.

We examine some results concerning independence.

Prop 3.2.1

Let T be stable. Let $\{\underline{a}_\beta : \beta < \alpha\}$ be a sequence of tuples in $\bar{M}$ such that for all $\beta < \alpha$ $tp(\underline{a}_\beta / (\bigcup\{\underline{a}_\beta : \beta < \alpha\} \cup A))$ does not fork over A. Then $\{\underline{a}_\beta : \beta < \alpha\}$ is independent over A.

This proposition can be proved by an easy induction argument using forking symmetry.

Prop 3.2.2

Let T be stable. Let $A \subseteq B \subseteq \bar{M}$ and let $\Sigma \in S_n(A)$ and $\Delta \in S_n(B)$ with $\Sigma \subseteq \Delta$. Let $M \preceq \bar{M}$ such that $A \subseteq M$ and M and B are independent over A. Then Δ does not fork over A iff Σ has an extension $\Delta' \in S_n(M)$ such that the heir of Δ' over $M \cup B$ is an extension of Δ.

Proof:

$\Rightarrow$: Suppose Δ does not fork over A. Then by Thm 3.1.3 (d) Δ has a non-forking extension over $M \cup B$, say Γ. Then Γ does not fork over A and hence does not fork over M. By Thm 3.1.3 (a) Γ is the heir of $\Delta' = \Gamma|_M$. Δ' extends Σ and so the result is proved.

$\Leftarrow$: Let $\Delta' \in S_n(M)$ be such that the heir of Δ' over $M \cup B$, say Γ, is an extension of Δ. Let $\underline{a}$ be a tuple realizing Γ. Then $tp(\underline{a} / (M \cup B))$ does not fork over M. Now M and B are independent over A and so $tp(B/M)$ does not fork over A. Hence $tp(B/(M \cup \underline{a}))$ does not fork over A. Hence by Thm 3.1.3(c) $tp(B/(A \cup \underline{a}))$ does not fork over A. Then by forking symmetry we have that $tp(\underline{a}/B) = \Delta$ does not fork over A.

we mention the following corollary to the above proposition.

Corollary 3.2.3

Let T be stable and $A \subseteq B \subseteq \bar{M}$. Let $\Sigma \in S(A)$, $\Delta \in S(B)$ with $\Sigma \subseteq \Delta$. Then Δ does not fork over A iff for all $M \leq \bar{M}$ with $A \subseteq M$ there is an extension $\Delta' \in S(M)$ of Σ such that the heir of Δ' over $M \cup B$ is an extension of Δ.

The following proposition will be needed in the next section.

Prop 3.2.4

Let T be stable, $A \subseteq B \subseteq \bar{M}$ and $\Sigma(\underline{x}) \in S_n(B)$. Let $M \leq \bar{M}$ with $A \subseteq M$ and M, B independent over A. Then $\Sigma(\underline{x})$ does not fork over A iff $\Gamma(\underline{x}) = \Sigma(\underline{x}) \cup \Phi(\underline{x})$ is consistent, where $\Phi(\underline{x}) = \{\neg\varphi(\underline{x}, \underline{a}) : \underline{a}$ a tuple in $M \cup B$, $\varphi(\underline{x}, \underline{y}) \in L$ and $\varphi(\underline{x}, \underline{a})$ not satisfied in $M\}$.

Proof:

Let $\Sigma'(\underline{x}) \in S_n(M \cup B)$. Then we have that $\Phi(\underline{x}) \subseteq \Sigma'(\underline{x})$ iff $\Sigma'(\underline{x})$ is the co-heir of $\Sigma'|_M(\underline{x})$ iff $\Sigma'(\underline{x})$ is the heir of $\Sigma'|_M(\underline{x})$. Hence $\Gamma(\underline{x})$ is consistent iff $\Sigma(\underline{x})$ has an extension $\Sigma'(\underline{x})$ in $S_n(M \cup B)$ which is the heir of its restriction to M. Hence by Prop. 3.2.2 we have that $\Gamma(\underline{x})$ is consistent iff $\Sigma(\underline{x})$ does not fork over A.

We have used the concept of independence to aid our study of forking. We mention that in Model Theoretic Classification Theory the concept of independence is important in its own right since it provides a model-theoretic generalization of the concept of algebraic independence in Field Theory.

3. $S_n(A)$ as a topological space:

If $A \subseteq \overline{M}$ we can define a topology on the set $S_n(A)$ $(n<\omega)$. In this section we examine the implications of the topological structure of $S_n(A)$ to forking.

To begin with we define $F_n(A)$ (for $n<\omega$) as follows:

$$F_n(A) = \{\varphi \in L(A) : \varphi \text{ has free variables } \underline{x}\}$$

where $\underline{x}$ is an n-tuple of variables. As for types we are not concerned with the actual variables $\underline{x}$ ie if we replace $\underline{x}$ with another $\underline{y}$ to obtain $F_n'(A)$ we do not distinguish between $F_n(A)$ and $F_n'(A)$.

We define an equivalence relation on $F_n(A)$ as follows: If $\varphi(\underline{x}), \psi(\underline{x}) \in F_n(A)$ then $\varphi(\underline{x}) \sim \psi(\underline{x})$ iff $T \vdash (\forall \underline{x})(\varphi(\underline{x}) \Leftrightarrow \psi(\underline{x}))$.

We define $B_n(A)$ to be $F_n(A)/\sim$ ie the set of equivalence classes of $F_n(A)$ under $\sim$. We turn $B_n(A)$ into a Boolean algebra as follows:

If $\varphi(\underline{x}), \psi(\underline{x}) \in F_n(A)$ let $[\varphi(\underline{x})]$ and $[\psi(\underline{x})]$ denote their equivalence classes in $B_n(A)$. Define a partial order on $B_n(A)$ by $[\varphi(\underline{x})] \leq [\psi(\underline{x})]$ iff we have $T \vdash (\forall \underline{x})(\varphi(\underline{x}) \Rightarrow \psi(\underline{x}))$. (This is well-defined because of the way $\sim$ is defined.)

$\langle B_n(A), \leq \rangle$ is then a Boolean algebra with:

$$[\varphi(\underline{x})] \vee [\psi(\underline{x})] = [\varphi(\underline{x}) \vee \psi(\underline{x})]$$

$$[\varphi(\underline{x})] \wedge [\psi(\underline{x})] = [\varphi(\underline{x}) \wedge \psi(\underline{x})]$$

$$[\varphi(\underline{x})]' = [\neg\varphi(\underline{x})]$$

$$1 = [\varphi(\underline{x}) \vee \neg\varphi(\underline{x})] \quad , \text{ and}$$

$$0 = [\varphi(\underline{x}) \wedge \neg\varphi(\underline{x})] \ .$$

Recall that if B is a Boolean algebra then the Stone space of B, $\underline{\sigma(B)}$ is the topological space defined as follows:

Let $\sigma(B)$ be the set of all ultrafilters in B.

For all $a \in B$ put $u(a) = \{\mathcal{U} \in \sigma(B) : a \in \mathcal{U}\}$

Then $\{u(a) : a \in B\}$ is a base for the topology of $\underline{\sigma(B)}$.

Now let $p : Fn(A) \twoheadrightarrow Bn(A)$ be the canonical surjection given by $p(\varphi(\underline{x})) = [\varphi(\underline{x})]$. Then p determines a surjection $\bar{p} : Sn(A) \twoheadrightarrow \sigma(Bn(A))$ given by:

If $\Sigma \in Sn(A)$ then $\bar{p}(\Sigma) = p[\Sigma] = \{p(\varphi) : \varphi \in \Sigma\}$

$$= \{[\varphi] : \varphi \in \Sigma\}.$$

We can now turn $Sn(A)$ into a topological space by letting $Sn(A)$ have the initial topology w.r.t. the map $\bar{p} : Sn(A) \twoheadrightarrow \sigma(Bn(A))$.

We now examine some results concerning the topology on $Sn(B)$.

For all $\varphi \in Fn(A)$ put $\langle\varphi\rangle = \{\Sigma \in Sn(A) : \varphi \in \Sigma\}$. Then we have:

<u>Prop 3.3.1</u> (own proof)

Let $A \subseteq \overline{M}$. Then $\{\langle\varphi\rangle : \varphi \in Fn(A)\}$ is a base for the topology on $Sn(A)$.

<u>Proof:</u>

Let X be an open subset of $Sn(A)$ and let $\Delta \in X$. Then there

is a base element $\mathcal{U}([\varphi])$ of $\sigma(B_n(A))$ such that: $\Delta \in \bar{p}^{-1}[\mathcal{U}([\varphi])] \subseteq X$. Now we have that:

$$\begin{aligned}\mathcal{U}([\varphi]) &= \{\mathcal{U} \in \sigma(B_n(A)) : [\varphi] \in \mathcal{U}\} \\ &= \{\bar{p}(\Sigma) : \Sigma \in S_n(A), \varphi \in \Sigma\} \\ &= \bar{p}[\{\Sigma \in S_n(A) : \varphi \in \Sigma\}] = \bar{p}[\langle\varphi\rangle].\end{aligned}$$

Then $\langle\varphi\rangle \subseteq \bar{p}^{-1}[\mathcal{U}([\varphi])]$. Now if $\Sigma \in \bar{p}^{-1}[\mathcal{U}([\varphi])]$ we have $\bar{p}(\Sigma) \in \mathcal{U}([\varphi])$, hence $[\varphi] \in \bar{p}(\Sigma) = p[\Sigma]$. Then there is $\psi \in \Sigma$ such that $[\varphi] = p(\psi) = [\psi]$. Then $\varphi \sim \psi$ ie $T \vdash (\forall \underline{x})(\varphi(\underline{x}) \Leftrightarrow \psi(\underline{x}))$. Then since Σ is a (complete and consistent) type of $Th(\bar{M}, a)_{a \in A}$ we must have $\varphi \in \Sigma$. Hence $\Sigma \in \langle\varphi\rangle$ and so $\langle\varphi\rangle = \bar{p}^{-1}[\mathcal{U}([\varphi])]$. But then $\Delta \in \langle\varphi\rangle \subseteq X$. ∎

Recall that $\Sigma \in S_n(A)$ is a <u>principal type</u> generated by $\psi(\underline{x}) \in L(A)$ iff $\Sigma = \{\varphi(\underline{x}) \in L(A) : T \vdash (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}))\}$. We have the following characterization of principal types:

<u>Prop 3.3.2</u> (own proof)

Let $\Sigma \in S_n(A)$ $(A \subseteq \bar{M})$. Then Σ is principal iff Σ is an isolated point in <u>$S_n(A)$</u> (ie in $S_n(A)$ as a topological space.)

<u>Proof:</u>

Let Σ be principal with generator $\psi(\underline{x})$. Then we have that $\Sigma = \{\varphi(\underline{x}) \in L(A) : T \vdash (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}))\}$, and so $\Sigma = \{\varphi(\underline{x}) \in L(A) : [\psi(\underline{x})] \leq [\varphi(\underline{x})]\}$. Then we have $\bar{p}(\Sigma) = \{[\varphi(\underline{x})] : [\psi(\underline{x})] \leq [\varphi(\underline{x})]\}$. Hence $\bar{p}(\Sigma)$ is a principal ultrafilter in $B_n(A)$, and so is an isolated point in the Stone space <u>$\sigma(B_n(A))$</u> ie $\{\bar{p}(\Sigma)\}$ is open in <u>$\sigma(B_n(A))$</u>. By completeness of Σ we have $\Sigma = p^{-1}[p[\Sigma]]$ and so $\{\Sigma\} = \bar{p}^{-1}[\{\bar{p}(\Sigma)\}]$. Hence Σ is an isolated point of <u>$S_n(A)$</u>.

conversely let Σ be an isolated point of $\underline{S_n(A)}$. Then there is a base element $u([\psi])$ such that $\{\Sigma\} = \bar{p}^{-1}[u([\psi])]$. Then $u([\psi]) = \bar{p}[\{\Sigma\}] = \{\bar{p}(\Sigma)\}$ a singleton. Hence there is only one ultrafilter, $\bar{p}(\Sigma)$, containing $[\psi]$. Hence $[\psi]$ is an atom and $\bar{p}(\Sigma) = \{[\varphi(\underline{x})] : [\psi(\underline{x})] \leq [\varphi(\underline{x})]\}$. Then by completeness $\Sigma = \{\varphi(\underline{x}) \in L(A) : [\psi(\underline{x})] \leq [\varphi(\underline{x})]\} = \{\varphi(\underline{x}) \in L(A) : T \vdash (\forall \underline{x})(\psi(\underline{x}) \Rightarrow \varphi(\underline{x}))\}$ and so Σ is principal.

∎

Let T be stable and let $A \subseteq B \subseteq \bar{M}$. Let $X_n(A,B)$ be the set of all types $\Sigma(\underline{x}) \in S_n(B)$ which fork over A. The main result of this section is to show that $X_n(A,B)$ is open in $\underline{S_n(B)}$.

<u>Thm 3.3.3</u>

Let T be stable and $A \subseteq B \subseteq \bar{M}$. Then $X_n(A,B)$ is an open set in the topological space $\underline{S_n(B)}$.

<u>Proof:</u>

Let $M \preceq \bar{M}$ with $A \subseteq M$ and M, B independent over A. Let $\Phi(\underline{x})$ be as in Prop 3.2.4. Let $\Sigma(\underline{x}) \in X_n(A,B)$. Then by Prop 3.2.4 we have that $\Sigma(\underline{x}) \cup \Phi(\underline{x})$ is not consistent. Hence there is a $\psi(\underline{x}) \in \Sigma(\underline{x})$ such that $\Phi(\underline{x}) \cup \{\psi(\underline{x})\}$ is inconsistent. Consider the base element $\langle \psi(\underline{x}) \rangle$ of $\underline{S_n(B)}$. Let $\Delta(\underline{x}) \in \langle \psi(\underline{x}) \rangle$, then $\Delta(\underline{x}) \cup \Phi(\underline{x})$ is not consistent and hence by Prop 3.2.4. $\Delta(\underline{x}) \in X_n(A,B)$. Hence we have that $\Sigma(\underline{x}) \in \langle \psi(\underline{x}) \rangle \subseteq X_n(A,B)$. $X_n(A,B)$ is thus open.

∎

we have the following corollary:

<u>Corollary 3.3.4</u> (Pillay [83], own proof given)

Let T be stable, $A \subseteq B \subseteq \bar{M}$, $\Sigma(\underline{x}) \in S_n(A)$ and $\Sigma'(\underline{x}) \in S_n(B)$ a forking extension of $\Sigma(\underline{x})$. Then there is a formula $\psi(\underline{x}, \underline{y})$

in $cl(\Sigma')$ such that for all $C \subseteq \bar{M}$ with $A \subseteq C$, and for all $\Delta(\underline{x}) \in S_n(C)$ extending $\Sigma(\underline{x})$ we have that, $\psi(\underline{x}, \underline{y}) \in cl(\Delta)$ implies that $\Delta(\underline{x})$ forks over A.

Proof:

We have $\Sigma'(\underline{x}) \in X_n(A, B)$ which is open in $S_n(B)$ by Thm 3.3.3. Hence there is a $\psi(\underline{x}, \underline{y}) \in L$ and a tuple $\underline{b}$ in B such that $\Sigma'(\underline{x}) \in \langle \psi(\underline{x}, \underline{b}) \rangle \subseteq X_n(A, B)$. Now $\psi(\underline{x}, \underline{b}) \in \Sigma'(\underline{x})$ and so $\psi(\underline{x}, \underline{y}) \in cl(\Sigma')$. Let $C \subseteq \bar{M}$ with $A \subseteq C$. Then if $\Delta(\underline{x}) \in S_n(C)$ with $\psi(\underline{x}, \underline{y}) \in cl(\Delta)$ there is a tuple $\underline{c}$ in C such that $\psi(\underline{x}, \underline{c}) \in \Delta(\underline{x})$. Then by forking symmetry we have $\Delta(\underline{x}) \in X_n(A, C)$ ie $\Delta(\underline{x})$ forks over A.

∎

We can use this corollary to prove the following result.

Prop 3.3.5 (Pillay [83])

Let T be stable, $A \subseteq B \subseteq \bar{M}$, $\Sigma(\underline{x}) \in S(A)$, $\Delta(\underline{x}) \in S(B)$ with $\Sigma(\underline{x}) \subseteq \Delta(\underline{x})$. Then $\Delta(\underline{x})$ does not fork over A iff $cl(\Delta) \subseteq \beta(\Sigma)$.

Proof:

$\Rightarrow$: If $\Delta(\underline{x})$ does not fork over A then we have $\beta(\Sigma) = \beta(\Delta)$. But $cl(\Delta) \subseteq \beta(\Delta)$ and so the result follows.

$\Leftarrow$: we prove the result by contraposition so assume that $\Delta(\underline{x})$ forks over A. Let $\psi(\underline{x}, \underline{y})$ be the formula in $cl(\Delta)$ given by corollary 3.3.4. Let $M \preceq \bar{M}$ and let $\Gamma(\underline{x}) \in S(M)$ be a non-forking extension of $\Sigma(\underline{x})$. (The existence of $\Gamma(\underline{x})$ is given by Thm 3.1.3 (d.)). Then we have $\psi(\underline{x}, \underline{y}) \notin cl(\Gamma)$ and since $\beta(\Sigma) \subseteq cl(\Gamma)$ we have $\psi(\underline{x}, \underline{y}) \notin \beta(\Sigma)$. Hence $cl(\Delta) \not\subseteq \beta(\Sigma)$.

∎

4. The cardinality of the set of Non-forking Types.

In this section we prove an important result concerning the cardinality of the set of non-forking extensions of a type. First we need a lemma:

Lemma 3.4.1

Let T be stable, $B \subseteq \overline{M}$ and $\Sigma \in S(B)$. Then there is an $A \subseteq B$ such that A is countable and Σ does not fork over A.

Proof:

We prove the result by contradiction, so assume that it is false. We use induction to define a chain $\{A_\alpha\}_{\alpha < \omega_1}$ of subsets of B such that for all $\alpha < \omega_1$, $\Sigma|_{A_{\alpha+1}}$ forks over A_α. Let $A_0 \subseteq B$ be countable. Let $\delta < \omega_1$ and suppose we have found A_α for all $\alpha < \delta$. If δ is a limit ordinal let $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. If $\delta = \beta + 1$ then A_β is countable and so Σ must fork over A_β. By Prop. 3.1.6 there is a tuple $\underline{b}$ in B such that $\Sigma|_{A_\beta \cup \underline{b}}$ forks over A_β. Let $A_\delta = A_\beta \cup \underline{b}$. Thus the induction is complete. Now if $\alpha < \beta < \omega_1$ then $\beta(\Sigma|A_\alpha) \not\leq \beta(\Sigma|A_\beta)$, but this contradicts the fact that we are working with a countable language. ∎

We can now prove the desired result.

Thm 3.4.2

Let T be stable, $A \subseteq B \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S_n(A)$. Then $\Sigma(\underline{x})$ has at most $2^{\aleph_0}$ non-forking extensions in $S_n(B)$.

Proof:

Let $M \preceq \overline{M}$ with $A \subseteq M$. By lemma 3.4.1 there is a $C \subseteq A$ such that C is countable and $\Sigma(\underline{x})$ does not fork over C. Let $N \prec M$ such that $C \subseteq N$ and let $\Delta(\underline{x}) \in S_n(M)$ be a non-forking extension of $\Sigma(\underline{x})$. Then $\Delta(\underline{x})$ does not fork over A and hence does not fork over N. Hence $\Delta(\underline{x})$ is the unique heir of $\Delta|_N(\underline{x})$. Now $|S(N)| \leq 2^{\aleph_0}$, and so the number of possible non-forking extensions of $\Sigma(\underline{x})$ over M is at most $2^{\aleph_0}$. Now if B is a model the result is proved. If B is not a model we can find a model $M \preceq \overline{M}$ such that $B \subseteq M$. Then any non-forking extension of $\Sigma(\underline{x})$ to $S(B)$ extends to a non-forking extension in $S(M)$ and the result follows.

$\blacksquare$

The above result will allow us to define the concepts of multiplicity and stationary types which we investigate in the next section.

5. Multiplicity and Stationary Types:

Let T be stable, $A \subseteq \overline{M}$ and let $\Sigma(\underline{x}) \in S_n(A)$. For all $B \subseteq \overline{M}$ such that $A \subseteq B$ let $N(B)$ be the set of all non-forking extensions of $\Sigma(\underline{x})$ in $S_n(B)$. Then we define the <u>multiplicity</u> of $\Sigma(\underline{x})$ to be the cardinal:

$$\mathrm{mult}(\Sigma) = \sup\{|N(B)| : A \subseteq B \subseteq \overline{M}\}$$

If $\varphi(\underline{x}, \underline{y})$ is a formula in L then we define the <u>φ-multiplicity</u> of $\Sigma(\underline{x})$ to be:

$$\mathrm{mult}_\varphi(\Sigma) = \sup\{|\{\Delta|_\varphi(\underline{x}) : \Delta(\underline{x}) \in N(B)\}| : A \subseteq B \subseteq \overline{M}\}$$

where $\Delta|_\varphi(\underline{x}) = \Delta(\underline{x}) \cap \{\varphi(\underline{x}, \underline{a}), \neg\varphi(\underline{x}, \underline{b}) : \underline{a}, \underline{b} \text{ tuples in } B\}$

By Thm 3.4.2 $\mathrm{mult}(\Sigma)$ and $\mathrm{mult}_\varphi(\Sigma)$ both exist and are both

less than or equal to $2^{\aleph_0}$.

We say that $\Sigma(\underline{x})$ is <u>stationary</u> iff $\mathrm{mult}(\Sigma) = 1$.
Thus $\Sigma(\underline{x})$ is stationary iff for any $B \subseteq \overline{M}$ such that $A \subseteq B$, $\Sigma(\underline{x})$ has exactly one non-forking extension over B.

For example let T be $Th(X)$ where X is an infinite set (with no relations or functions except equality). T is ω-stable. There are only two classes of 1-types over models of T, viz. β_1 which contains the formula $(x=y)$ and β_2 which does not. $\beta_1 \subseteq \beta_2$. If A is any subset of $\overline{M}$ and $\Sigma(x) \in S_1(A)$, then there are two possibilities:
i) For some $a \in A$ $(x=a) \in \Sigma(x)$
ii) $\Sigma(x) = \{\neg(x=a) : a \in A\}$
For (i) we have $\beta(\Sigma) = \beta_1$ and for all $B \supseteq A$, $\Sigma(\underline{x})$ has only one extension to B, viz. $tp(a/B)$, which must be a non-forking extension. For (ii) we have $\beta(\Sigma) = \beta_2$ and for all $B \supseteq A$, $\Sigma(x)$ has exactly one non-forking extension to B viz. $\{\neg(x=b) : b \in B\}$. Hence $\mathrm{mult}(\Sigma) = 1$ ie Σ is stationary. In fact an inductive argument shows all types of T are stationary.

Note also that if T is any stable theory and $M \preceq \overline{M}$, then by Thm 3.1.3. any $\Sigma \in S(M)$ has only one non-forking extension over any $A \supseteq M$ (viz. its heir). Hence all $\Sigma \in S(M)$ are stationary.

However it is not the case that all types are stationary. Let L be a language containing binary predicate symbols E_n for all $n<\omega$. Let T be the theory which says that each E_n $(n<\omega)$ is an equivalence relation such that E_0 has two infinite blocks, and

E_{n+1} partitions each block of E_n into two infinite blocks. T is stable. $O_1(T)$ has only two elements, β_1 which contains the formula $(x=y)$ and β_2 which does not. In any model each E_n has 2^{n+1} blocks, and so for each $n<\omega$ the formula $E(x,y)$ is represented in every type over a model of T. Let M be the countable model of T. If $(a_n)_{n<\omega}$ is a sequence in M we say that it is a strict Cauchy sequence in M if $M \models E_m[a_n, a_{n+1}]$ iff $m \leq n$ for all $m,n<\omega$. If $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ are two strict Cauchy sequences in M we say that they are equivalent iff $M \models E_m(a_n, b_n)$ for all $m,n<\omega$. Now given a strict cauchy sequence $\bar{a} = (a_n)_{n<\omega}$ in M, let $\Delta\bar{a} = \{E_m(x, a_n) : m,n<\omega\}$. $\Delta\bar{a}$ is consistent. Since M is countable there are $2^{\aleph_0}$ non-equivalent strict Cauchy sequences in M. We thus obtain $2^{\aleph_0}$ corresponding 1-types over M. (Hence T is not ω-stable). There is a unique type $\Sigma(x) \in S_1(T)$. $\beta(\Sigma) = \beta_2$. For each strict Cauchy sequence $\bar{a}$ let $\Sigma\bar{a}$ be the 1-type in $S_1(M)$ determined by $\Sigma\bar{a} \cup \{\neg(x=b) : b \in M\}$. Each $\Sigma\bar{a}$ is a non-forking extension of $\Sigma(x)$ over M. Thus we have $\mathrm{mult}(\Sigma) \geq 2^{\aleph_0}$ and so $\mathrm{mult}(\Sigma) = 2^{\aleph_0}$, the maximum possible value.

It is also possible for a type Σ to have $1 < \mathrm{mult}(\Sigma) < 2^{\aleph_0}$. For example let L be a language with a single binary predicate symbol E and let T be the theory which says that E is an equivalence relation with two infinite blocks. T is ω-stable. As before $O_1(T)$ has two elements, β_1 containing $(x=y)$ and β_2 which does not contain $(x=y)$. Let M be a model of T and let $a,b \in M$ such that $M \models \neg E(a,b)$. Then $M \models (\forall x)(E(x,a) \vee E(x,b))$ and so any $\Sigma(x) \in S_1(M)$ must contain either $E(x,a)$ or $E(x,b)$. T has a unique 1-type $\Delta(x)$ over $\emptyset$. Then $\beta(\Delta) = \beta_2$. $\Delta(x)$ has two non-forking extensions over

M viz. $\Delta_1(x)$ determined by $\{E(x,a)\} \cup \{\neg(x=b) : b \in M\}$ and $\Delta_2(x)$ determined by $\{E(x,b)\} \cup \{\neg(x=b) : b \in M\}$. Hence $\mathrm{mult}(\Delta) = 2$. Later we will see that multiplicity can only be finite or $2^{\aleph_0}$.

We now investigate some results concerning multiplicity and stationary types.

<u>Lemma 3.5.1</u>

Let T be stable, $A \subseteq B \subseteq \bar{M}$ and $\Sigma_1(\underline{x}), \Sigma_2(\underline{x}) \in S_n(B)$ be types which do not fork over A. Let $M \preceq \bar{M}$ with $A \subseteq M$. Let $\Sigma_1'(\underline{x}), \Sigma_2'(\underline{x}) \in S_n(B \cup M)$ be non-forking extensions of $\Sigma_1(\underline{x})$ and $\Sigma_2(\underline{x})$ respectively, and let $\Delta_1(\underline{x}), \Delta_2(\underline{x}) \in S_n(M)$ be restrictions of $\Sigma_1'(\underline{x})$ and $\Sigma_2'(\underline{x})$ respectively. Let $\varphi(\underline{x},\underline{y})$ be in L. Then $\Delta_1|\varphi = \Delta_2|\varphi$ implies that $\Sigma_1|\varphi = \Sigma_2|\varphi$.

<u>Proof:</u>

For $i=1,2$ $\Sigma_i'(\underline{x})$ is a non-forking extension of $\Delta_i(\underline{x})$. Hence $\Sigma_i'(\underline{x})$ is the heir of $\Delta_i(\underline{x})$. Suppose that $\Delta_1|\varphi = \Delta_2|\varphi$. Let d_i be a defining schema for $\Delta_i(\underline{x})$ $(i=1,2)$. Let $\psi(\underline{y}) \in L(M)$ be such that $d_1\varphi = d_2\varphi = \psi(\underline{y})$. For $i=1,2$ $\Sigma_i'(\underline{x}) = d_i(B \cup M)$ and so for all tuples $\underline{a}$ in $B \cup M$ we have $\varphi(\underline{x},\underline{a}) \in \Sigma_1'(\underline{x})$ iff $\models \psi[\underline{a}]$ iff $\varphi(\underline{x},\underline{a}) \in \Sigma_2'(\underline{x})$. Hence $\Sigma_1'|\varphi = \Sigma_2'|\varphi$ and so $\Sigma_1|\varphi = \Sigma_2|\varphi$.

∎

<u>Prop 3.5.2</u>

Let T be stable and $A \subseteq M \preceq \bar{M}$. Let $\Sigma(\underline{x}) \in S(A)$. Then $\mathrm{mult}(\Sigma)$ is equal to the number of non-forking extensions of $\Sigma(\underline{x})$ to M. If $\varphi(\underline{x},\underline{y}) \in L$ then $\mathrm{mult}_\varphi(\Sigma)$ is equal to the number of non-forking extensions of $\Sigma(\underline{x})$ to M restricted to $\varphi(\underline{x},\underline{y})$.

(This result follows immediately from Lemma 3.5.1 if we state its

conclusion in the contrapositive.) ∎

<u>Thm 3.5.3</u>

Let T be stable, $A \subseteq \overline{M}$, $\Sigma(\underline{x}) \in S(A)$ and $\varphi(\underline{x}, \underline{y}) \in L$. Then $\mathrm{mult}_\varphi(\Sigma)$ is finite.

<u>Proof:</u>

Suppose $\mathrm{mult}_\varphi(\Sigma)$ is infinite. By Prop. 3.5.2 there is $M \preceq \overline{M}$ with $A \subseteq M$ such that $\Sigma(\underline{x})$ has infinitely many restrictions of types to $\varphi(\underline{x}, \underline{y})$ which are non-forking extensions to M. Add a new predicate symbol P to L. Let α be a cardinal. Let Σ be the set of formulae which contains the following:

(1.) "P is an elementary substructure of $\overline{M}$"

(2.) $\Sigma(\underline{x}_\beta)$ for all (ordinals) $\beta < \alpha$

(3.) $\{ (\forall \underline{z})(P(\underline{z}) \Rightarrow \neg\psi(\underline{x}_\beta, \underline{z})) : \beta < \alpha, \psi(\underline{x}, \underline{z}) \notin \beta(\Sigma) \}$.

(4.) $P(\underline{y}_{\beta,\gamma})$ for all $\beta < \gamma < \alpha$.

(5.) $\varphi(\underline{x}_\beta, \underline{y}_{\beta,\gamma}) \Longleftrightarrow \neg\varphi(\underline{x}_\gamma, \underline{y}_{\beta,\gamma})$ for all $\beta < \gamma < \alpha$

Now any finite subset can be satisfied as follows: Let P be M, let the $\underline{x}_\beta$'s realize non-forking extensions of $\Sigma(\underline{x})$ to M with pairwise distinct restrictions to $\varphi(\underline{x}, \underline{y})$ and let the $\underline{y}_{\beta,\gamma}$'s be interpreted by tuples in M which serve to distinguish these restrictions to φ. Hence Σ is consistent and so has a model $N \succeq A$. Then there are α (distinct) non-forking extensions of $\Sigma(\underline{x})$ to N. Choosing $\alpha > 2^{\aleph_0}$ contradicts Thm 3.4.2. Hence $\mathrm{mult}_\varphi(\Sigma)$ is finite. ∎

We can relate the concept of stationarity to the rank R^φ which was introduced in Chapter 2. First we need a lemma which for the sake of brevity we do not prove. (The proof is straightforward and uses Thm 3.1.3 and Prop 3.2.2.)

Lemma 3.5.4.

Let T be stable and $A \subseteq M \preceq \bar{M}$. Let $\Sigma \in S(A)$ and let $\Sigma \subseteq \Delta \in S(M)$. Assume M is $(|A| + \aleph_0)^+$-saturated. Then Δ does not fork over A iff for all models N such that $A \subseteq N \preceq M$, Δ is the heir of $\Delta|_N$.

Thm 3.5.5

Let T be stable, $A \subseteq \bar{M}$ and $\Sigma(\underline{x}) \in S_n(A)$. The following are equivalent:

a) $\Sigma(\underline{x})$ is stationary

b) For all $B \supseteq A$, there is $\Delta(\underline{x}) \in S_n(B)$ with $\Sigma(\underline{x}) \subseteq \Delta(\underline{x})$ such that $R^\varphi(\Sigma) = R^\varphi(\Delta)$ for all $\varphi(\underline{x}, \underline{y}) \in L$.

c) There is a model $M \supseteq A$ and $\Delta(\underline{x}) \in S_n(M)$ extending $\Sigma(\underline{x})$ such that $R^\varphi(\Sigma) = R^\varphi(\Delta)$ for all $\varphi(\underline{x}, \underline{y}) \in L$.

Proof: (Sketch)

(b.) $\Leftrightarrow$ (c.):

(b.) $\Rightarrow$ (c.) is obvious. Assume (c.) and let $B \supseteq A$. $\Delta(\underline{x})$ has an extension $\Delta'(\underline{x}) \in S_n(M \cup B)$ such that $R^\varphi(\Delta') = R^\varphi(\Delta)$ for all $\varphi(\underline{x}, \underline{y}) \in L$. Then $R^\varphi(\Delta') = R^\varphi(\Sigma)$ for all $\varphi(\underline{x}, \underline{y}) \in L$. Let $\Gamma(\underline{x}) = \Delta'|_B(\underline{x})$. Then by Prop 2.2.1 we have $R^\varphi(\Sigma) = R^\varphi(\Gamma)$ for all $\varphi(\underline{x}, \underline{y}) \in L$.

(a) $\Rightarrow$ (b.):

If $\Sigma(\underline{x})$ is a type, $\varphi(\underline{x}, \underline{y}) \in L$ and $m < \omega$ then $R^\varphi(\Sigma) \geq m$ iff $\Gamma(\Sigma, \varphi, m)$ is consistent where:

$$\Gamma(\Sigma, \varphi, m) = \bigcup\{\Sigma(\underline{x}_f) : f \in {}^m 2\} \cup \{\varphi(\underline{x}_f, \underline{y}_{f|_i}) : f \in {}^m 2, i < m, f(i) = 0\}$$
$$\cup \{\varphi(\underline{x}_f, \underline{y}_{f|_i}) : f \in {}^m 2, i < m, f(i) = 1\}.$$

Assume (a.). Let $B \supseteq A$ and let $\Sigma'(\underline{x})$ be the unique non-forking extension of $\Sigma(\underline{x})$ over B. Let $\varphi(\underline{x}, \underline{y}) \in L$. Let $m < \omega$ be such that

$R^\varphi(\Sigma) = m$. Then $R^\varphi(\Sigma') \leq m$. (see prop 2.2.1, Thm 2.2.3). Then we have that $\Gamma(\Sigma, \varphi, m)$ is consistent. Let $\underline{a}_f$, $f \in {}^m 2$ and $\underline{b}_g$, $g \in {}^{m>}2$ be tuples realizing $\Gamma(\Sigma, \varphi, m)$. Thus we have:

$$\models \bigwedge_{f \in {}^m 2} \Sigma(\underline{a}_f) \wedge \bigwedge_{\substack{f \in {}^m 2 \\ f(i)=0}} \varphi(\underline{a}_f, \underline{b}_{f|i}) \wedge \bigwedge_{\substack{f \in {}^m 2 \\ f(i)=1}} \neg\varphi(\underline{a}_f, \underline{b}_{f|i}).$$

Let c be the set of all elements in the tuples $\underline{a}_f$, $f \in {}^m 2$ and $\underline{b}_g$, $g \in {}^{m>}2$. Let $\Theta = tp(c/A)$. Let Θ' be a non-forking extension of Θ in $S(B)$. Let $\underline{c}_f$, $f \in {}^m 2$ and $\underline{d}_g$, $g \in {}^{m>}2$ be tuples realizing Θ'. Then for all $f \in {}^m 2$, $tp(\underline{c}_f/B)$ does not fork over A. But for $f \in {}^m 2$ $tp(\underline{c}_f/A) = tp(\underline{a}_f/A) = \Sigma(\underline{x})$, and so $tp(\underline{c}_f/B) = \Sigma'(\underline{x})$. Hence $\Gamma(\Sigma', \varphi, m)$ is consistent since it is realized by $\underline{c}_f$, $f \in {}^m 2$ and $\underline{d}_g$, $g \in {}^{m>}2$. Hence $R^\varphi(\Sigma') \geq m$. Hence $R^\varphi(\Sigma') = m = R^\varphi(\Sigma)$.

(b) $\Rightarrow$ (a.):

Let $B \supseteq A$ and let $M \supseteq B$ be an $(|A| + \aleph_0)^+$-saturated model. By Prop 3.5.2 it suffices to show that $\Sigma(\underline{x})$ has a unique non-forking extension in $S_n(M)$. Let $\Sigma'(\underline{x}) \in S_n(M)$ be such that $R^\varphi(\Sigma') = R^\varphi(\Sigma)$ for all $\varphi(\underline{x}, \underline{y}) \in L$. Let d be the defining schema for $\Sigma(\underline{x})$ given by: for all $\varphi(\underline{x}, \underline{y}) \in L$ $d\varphi(\underline{y})$ is the formula which says "$R^\varphi(\psi(\underline{x}) \wedge \varphi(\underline{x}, \underline{y})) \geq n$" where $\psi(\underline{x}) \in \Sigma(\underline{x})$ is such that $R^\varphi(\Sigma) = R^\varphi(\psi) = n_\varphi$ say. Then d is also a defining schema for $\Sigma'(\underline{x})$. We wish to show that $\Sigma'(\underline{x})$ does not fork over A. For this let N be a model such that $A \subseteq N \leq M$. Since $\Sigma'(\underline{x})$ is definable over A it is definable over N. Hence $\Sigma'(\underline{x})$ is the heir of $\Sigma'|_N(\underline{x})$. By lemma 3.5.4 we then have that $\Sigma'(\underline{x})$ does not fork over A. We now wish to show that if $\Delta(\underline{x}) \in S_n(M)$ is a non-forking extension of $\Sigma'(\underline{x})$ then $R^\varphi(\Delta) = n_\varphi$ for all $\varphi(\underline{x}, \underline{y}) \in L$. Suppose this is false. Then for some $\varphi(\underline{x}, \underline{y}) \in L$ we have $R^\varphi(\Delta) = k < n_\varphi$. Hence for some

$\sigma(\underline{x}, \underline{a}) \in \Delta(\underline{x})$ where $\sigma(\underline{x}, \underline{z}) \in L$, we have $R^{\varphi}(\sigma(\underline{x}, \underline{a})) = k$. Hence the formula saying "$\sigma(\underline{x}, \underline{z})$ and $R^{\varphi}(\sigma(\underline{x}, a)) \leq k$" is in $cl(\Delta)$. But $cl(\Delta) = \beta(\Sigma) = cl(\Sigma')$. Hence there is a tuple $\underline{c}$ in N such that $R^{\varphi}(\sigma(\underline{x}, \underline{c})) \leq k$ and $\sigma(\underline{x}, \underline{c}) \in \Sigma'(\underline{x})$. This contradicts the fact that $R^{\varphi}(\Sigma') = n_{\varphi}$ and so the claim is true. Hence we have that if $\Delta(\underline{x}) \in S(M)$ is a non-forking extension of $\Sigma(\underline{x})$ then for all $\varphi(\underline{x}, \underline{y}) \in L$, $R^{\varphi}(\Delta) = R^{\varphi}(\Sigma') = R^{\varphi}(\Sigma) < \omega$. Hence $\Delta(\underline{x}) = \Sigma'(\underline{x})$. So we have shown that $\Sigma'(\underline{x})$ is the unique non-forking extension of $\Sigma(\underline{x})$. Hence $\Sigma(\underline{x})$ is stationary. ∎

6. Forking and Definability:

Definitions:

Let T be stable.

If $A \subseteq M$ and $\Sigma(\underline{x}) \in S_n(A)$ with defining schema d, we say that d is a <u>good</u> defining schema iff the following hold:

a) For each $\{\varphi_i(\underline{x}, \underline{y}_i) : i < m\} \subseteq L$ with $m < \omega$ we have

$$\models (\forall \underline{y}_0 \cdots \underline{y}_{m-1})\Big(\bigwedge_{i<m} d\varphi_i(\underline{y}_i) \Rightarrow (\exists \underline{x})\big(\bigwedge_{i<m} \varphi_i(\underline{x}, \underline{y}_i)\big)\Big)$$

and b) For all $\varphi(\underline{x}, \underline{y}) \in L$, $\models (\forall \underline{y})((d\neg\varphi)(\underline{y}) \Leftrightarrow \neg d\varphi(\underline{y}))$.

We say that $\Sigma(\underline{x})$ is <u>well-definable</u> iff it has a good defining schema over B (ie for all $\varphi(\underline{x}, \underline{y})$, $d\varphi(\underline{y}) \in L(B)$.)

We immediately have the following easy but important results:

Thm 3.6.1

Let T be stable, $A \subseteq \overline{M}$, $\Sigma(\underline{x}) \in S_n(A)$ and d a defining schema

for $\Sigma(\underline{x})$. Then:

a) The following are equivalent:

i) d is a good defining schema

ii) For all $B \supseteq A$, $d(B) \in S_n(B)$.

iii) There is $M \preceq \overline{M}$ with $A \subseteq M$ such that $d(M) \in S_n(M)$

b) If $A \preceq \overline{M}$ (ie A is in fact a model) then $\Sigma(\underline{x})$ is well-definable.

■

Let $\varphi(\underline{x}, \underline{y})$ be a formula in $L(A)$ where $\underline{x}$ and $\underline{y}$ are n-tuples of variables ($n < \omega$). If $\varphi(\underline{x}, \underline{y})$ defines an equivalence relation on $\overline{M}$ with finitely many blocks, then we say that $\varphi(\underline{x}, \underline{y})$ is an <u>n-ary finite equivalence relation over A.</u> We denote the set of all n-ary finite equivalence relations over A by $FE^n(A)$. We put $FE(A) = \bigcup_{n<\omega} FE^n(A)$ ie $FE(A)$ is the set of all finite equivalence relations over A. Note that "to be a finite equivalence relation with exactly m blocks" is a first order property. Also if $M \preceq \overline{M}$ with $A \subseteq M$ and $E \in FE^n(A)$, then each block of E has a non-empty intersection with M. (consider the first order sentence saying that there are m non-equivalent n-tuples.)

A formula $\varphi(\underline{x}, \underline{b})$ where $\underline{x}$ is an n-tuple of variables is said to be <u>almost over A</u> iff there is an $E \in FE^n(A)$ such that

$$\models (\forall \underline{x})(\forall \underline{y})(\varphi(\underline{x}, \underline{b}) \wedge E(\underline{x}, \underline{y}) \Rightarrow \varphi(\underline{y}, \underline{b})).$$

We then say that $\varphi(\underline{x}, \underline{b})$ <u>depends on E</u>.

Let $A \subseteq B \subseteq \overline{M}$. We say that $\Sigma(\underline{x}) \in S(B)$ is <u>almost over A</u> iff every formula in $\Sigma(\underline{x})$ is almost over A.

If $\varphi(\underline{x}) \in L(A)$ then $\varphi(\underline{x})$ depends on $(\varphi(\underline{y}) \Leftrightarrow \varphi(\underline{z})) \in FE(A)$.

($(\varphi(\underline{y}) \Leftrightarrow \varphi(\underline{z}))$ describes an equivalence relation with 2 blocks.)
Hence any $L(A)$ formula is over A.

Note that if $\varphi(\underline{x}, \underline{b})$ is almost over A and $M \preceq \overline{M}$ with $A \subseteq M$ then there is some tuple $\underline{a}$ in M such that $\models \varphi(\underline{a}, \underline{b})$ (There is a tuple $\underline{c}$ in $\overline{M}$ such that $\models \varphi(\underline{c}, \underline{b})$ and there is an E in $FE(A)$ such that $\varphi(\underline{x}, \underline{b})$ depends on E. Then there is an $\underline{a}$ in M with $\models E(\underline{c}, \underline{a})$, and so $\models \varphi(\underline{a}, \underline{b})$.)

If $A \subseteq B \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S(B)$ we say that $\Sigma(\underline{x})$ is <u>definable almost over A</u> iff for each formula $\varphi(\underline{x}, \underline{y}) \in L$ with $\underline{y}$ an m-tuple of variables, there is an $E_\varphi \in FE^m(A)$ such that for all m-tuples $\underline{b}_1$ and $\underline{b}_2$ in B we have that, $\models E_\varphi(\underline{b}_1, \underline{b}_2)$ implies : $\varphi(\underline{x}, \underline{b}_1) \in \Sigma(\underline{x})$ iff $\varphi(\underline{x}, \underline{b}_2) \in \Sigma(\underline{x})$. The mapping $\varphi(\underline{x}, \underline{y}) \mapsto E_\varphi$ is called a <u>weak defining schema over A</u> for $\Sigma(\underline{x})$.

For example consider the case where L has a single binary predicate symbol E. Let T be the theory that says that E is an equivalence relation with two infinite classes. Then T is stable.
Let M be a countable model of T and let $\Sigma(x)$ be the unique 1-type in $S_1(T)$. Let Σ_1, Σ_2 be the two non-forking extensions of Σ over M. (see ps 47). Then $\Sigma_1(x) \cup \Sigma_2(y) \vdash \neg E(x,y)$. (where x, y are distinct variables). $\Sigma_1(x)$ is determined by the formulae $\{E(x,a)\} \cup \{\neg(x=b) : b \in M\}$, for some $a \in M$. $\Sigma_1(x)$ is definable almost over A, since if $b_1, b_2 \in M$ and $\models (b_1, b_2)$ then $E(x, b_1) \in \Sigma_1$ iff $E(x, b_2) \in \Sigma_1(x)$. Also $(x = b) \in \Sigma_1(x)$ iff $b \neq b$ for all $b \in M$.

For another example consider the case where L is the language for groups and T is any stable theory in L. Let $M \preceq \overline{M}$ be definable by a formula in $L(A)$. Let $\varphi(x, \underline{b}) \in L(A)$ define a subgroup which

has finite index in M. Then $\varphi(x, \underline{b})$ is almost over A. To prove this consider the following: If $\varphi(x, \underline{y})$ is any formula in L then there is an $m < \omega$ such that if $\varphi(x, \underline{b}_i)$ for $i \in I$, define subgroups, then there is an $I_0 \subseteq I$ with $|I_0| \leq m$ such that $\cap\{G[\varphi(x, \underline{b}_i)] : i \in I\} = \cap\{G[\varphi(x, \underline{b}_i)] : i \in I_0\}$ where $G[\varphi(x, \underline{b}_i)]$ denotes the subgroup defined by $\varphi(x, \underline{b}_i)$ for $i \in I$. Now if $\varphi(x, \underline{b})$ is the formula we are concerned with there are tuples $\underline{b} = \underline{b}_1, \underline{b}_2, \ldots, \underline{b}_k$ $(k < \omega)$ such that $\varphi(x, \underline{b}_i)$ is a subgroup of index n in M for all $i = 1, \ldots, k$ (where n is the original finite index of $G[\varphi(x, \underline{b})]$ in M) and the intersection of all the subgroups of index n in M of the form $G[\varphi(x, \underline{c})]$ is exactly $N = G[\bigwedge_{i=1}^{k} \varphi(x, \underline{b}_i)]$. Let $\psi(\underline{y})$ be the formula in $L(A)$ which says that $\varphi(x, \underline{y})$ defines a subgroup of index n in M. Then N is also defined by the formula:

$$(\exists \underline{y}_1 \ldots \underline{y}_k)\Big(\bigwedge_{i=1}^{k} \psi(\underline{y}_i) \wedge \bigwedge_{i=1}^{k} \varphi(x, \underline{y}_i) \wedge (\forall z)\big(\psi(\underline{z}) \Rightarrow (\forall w)\big(\bigwedge_{i=1}^{k} \varphi(w, \underline{y}_i) \Rightarrow \varphi(w, \underline{z})\big)\big)\Big).$$

Now this formula is in $L(A)$. Hence N is defined over A, $N \subseteq G[\varphi(x, \underline{b})]$ and N has finite index in M. Let $E(x_1, x_2)$ say that x_1, x_2 are in M and x_1, x_2 are in the same coset of N. Then $E \in FE^1(A)$ and $\models (\varphi(x_1, \underline{b}) \wedge E(x_1, x_2)) \Rightarrow \varphi(x_2, \underline{b})$. Thus we have proved the claim.

We now investigate the connection between the above concepts and forking.

<u>Prop 3.6.2</u>

Let T be stable. $A \subseteq B \subseteq M$ and $\Sigma(\underline{x}) \in S(B)$. Then $\Sigma(\underline{x})$ is definable almost over A iff $\Sigma(\underline{x})$ is definable over B by a defining schema d such that for all $\varphi(\underline{x}, \underline{y}) \in L$ the formula, $d\varphi(\underline{y})$, is almost over A.

<u>Proof:</u>

$\Rightarrow$: For each $\varphi(\underline{x},\underline{y}) \in L$ choose an E_φ such that the mapping $\varphi(\underline{x},\underline{y}) \mapsto E_\varphi$ is a weak defining schema over A for $\Sigma(\underline{x})$. Let $\varphi(\underline{x},\underline{y}) \in L$ and let $\underline{b}_1, \ldots, \underline{b}_k$ be a maximal set of pairwise non-equivalent tuples (w.r.t E_φ) in B, such that for all $i = 1, \ldots, k$ $\varphi(\underline{x},\underline{b}_i) \in \Sigma(\underline{x})$. Put $d\varphi(\underline{y})$ equal to $\bigvee_{i=1}^{k} E_\varphi(\underline{y},\underline{b}_i)$. Then we have that $d\varphi(\underline{y})$ is almost over A depending on E_φ. We also have that if $\underline{a}$ is a tuple in A, then $\varphi(\underline{x},\underline{a}) \in \Sigma(\underline{x})$ iff $\models d\varphi[\underline{a}]$ ie d is a defining schema for $\Sigma(\underline{x})$.

$\Leftarrow$: Suppose d is a defining schema for $\Sigma(\underline{x})$ with $d\varphi(\underline{y})$ depending on $E_\varphi \in FE(A)$, for all $\varphi(\underline{x},\underline{y}) \in L$. Then if for all tuples $\underline{b}_1, \underline{b}_2$ in B we have that, $\models E_\varphi(\underline{b}_1,\underline{b}_2)$ implies that, $\varphi(\underline{x},\underline{b}_1) \in \Sigma(\underline{x})$ iff $\varphi(\underline{x},\underline{b}_1) \in \Sigma(\underline{x})$.

<u>corollary 3.6.3</u>

Let T be stable, $A \subseteq B \subseteq \overline{M}$ and $\Sigma \in S(B)$. If Σ is definable over A then it is definable almost over A.

(This follows immediately from Prop 3.6.2)

<u>Prop 3.6.4.</u>

Let T be stable, $A \subseteq B \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S(B)$. If $\Sigma(\underline{x})$ is almost over A then $\Sigma(\underline{x})$ does not fork over A.

<u>Proof:</u>

We prove the result by contradiction; assume that $\Sigma(\underline{x})$ is almost over A and that it forks over A. Let $M \preceq \overline{M}$ such that $A \subseteq M$ and M, B are independent over A. By Prop 3.2.4. there are formulae $\varphi_1(\underline{x}), \ldots, \varphi_k(\underline{x})$ in $L(M \cup B)$ such that for all $i = 1, \ldots, k$ $\varphi_i(\underline{x})$ is not satisfied in M and $\Sigma(\underline{x}) \vdash \bigvee_{i=1}^{k} \varphi_i(\underline{x})$. Then there is some formula $\psi(\underline{x}) \in \Sigma(\underline{x})$ such that $\psi(\underline{x}) \vdash \bigvee_{i=1}^{k} \varphi_i(\underline{x})$. Now

$\psi(\underline{x})$ is almost over A and hence it is satisfied in M. Hence at least one of the $\varphi_i(\underline{x})$'s is satisfied in M, a contradiction. ∎

Prop 3.6.5

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma(\underline{x}) \in S(M)$. If $\Sigma(\underline{x})$ is definable almost over A then $\Sigma(\underline{x})$ does not fork over A.

Proof:

Let $N \preceq \overline{M}$ with $A \subseteq N$. For all $\varphi(\underline{x}, \underline{y}) \in L$ choose an E_φ such that the mapping $\varphi(\underline{x}, \underline{y}) \mapsto E_\varphi$ is a weak defining schema for $\Sigma(\underline{x})$ over A. Let d be the defining schema for $\Sigma(\underline{x})$ obtained as in the proof of Prop. 3.6.2. Then since M is a model we have that $d(N \cup M) \in S(N \cup M)$. (By Prop 1.2.1). Put $\Sigma'(\underline{x}) = d(N \cup M)$. If $\varphi(\underline{x}, \underline{y}) \in L$ let $\underline{b}_1, \ldots, \underline{b}_k$ be a maximal set of pairwise non-equivalent tuples (w.r.t E_φ) in M. Each E_φ block intersects N, and so there are $\underline{b}'_1, \ldots, \underline{b}'_k$ in N such that for all $i = 1, \ldots, k$ $\models E_\varphi(\underline{b}_i, \underline{b}'_i)$. Let $d'\varphi(\underline{y})$ be the formula $\bigvee_{i=1}^{k} E_\varphi(\underline{y}, \underline{b}'_i)$ in $L(N)$. We then have $\models (\forall \underline{y})(d\varphi(\underline{y}) \Leftrightarrow d'\varphi(\underline{y}))$. We have thus obtained a defining schema for $\Sigma'(\underline{x})$ over N. Hence $\Sigma'(\underline{x})$ is the heir of $\Sigma'|_N(\underline{x})$. By corollary 3.2.3 $\Sigma(\underline{x})$ does not fork over A. ∎

We immediately have the following corollary:

Corollary 3.6.6

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma \in S(M)$. If Σ is definable over A then Σ does not fork over A.

We now examine the connection between definability and stationarity.

Prop 3.6.7

Let T be stable, $A \subseteq \overline{M}$ and $\Sigma \in S_n(A)$. Then Σ is stationary iff there is an $M \preceq \overline{M}$ with $A \subseteq M$ such that Σ has a unique non-forking extension in $S_n(M)$.

Proof:

$\Rightarrow$: This follows directly from Prop 3.5.2 and the definition of stationary types.

$\Leftarrow$: Let $B \supseteq A$ and suppose Σ has at least two distinct non-forking extensions to B say Σ_1 and Σ_2. Then any non-forking extensions Δ_1 and Δ_2 of Σ_1 and Σ_2 to $M \cup B$ would be distinct. Now $\Delta_1|_M$ and $\Delta_2|_M$ are non-forking extensions of Σ in $S_n(M)$ and so we have $\Delta_1|_M = \Delta_2|_M = \Delta$ say. Then Δ_1 and Δ_2 are non-forking extensions of Δ and are therefore both heirs of Δ over $M \cup B$. This contradicts the fact that Δ has a unique heir. Hence Σ has a unique non-forking extension to B ie Σ is stationary. ∎

Prop 3.6.7

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma(\underline{x}) \in S(M)$. If $\Sigma(\underline{x})$ is definable over A then $\Sigma|_A(\underline{x})$ is stationary.

Proof:

Let $\Sigma'(\underline{x})$ be any non-forking extension of $\Sigma|_A(\underline{x})$ M. Let d be a defining schema over A for $\Sigma(\underline{x})$. Then for each $\varphi(\underline{x}, \underline{y}) \in L$ the formula $\neg(\varphi(\underline{x}, \underline{y}) \Leftrightarrow d\varphi(\underline{y}))$ is not represented in $\Sigma(\underline{x})$ and so it cannot be represented in $\Sigma'(\underline{x})$. (Any formula in $L(A)$ will be represented in $\Sigma(\underline{x})$ iff it is represented in $\Sigma'(\underline{x})$ since $\Sigma(\underline{x})$, $\Sigma'(\underline{x})$ are both extensions of $\Sigma|_A(\underline{x})$ to M.) Hence we have $\Sigma'(\underline{x}) = d(M) = \Sigma(\underline{x})$. So $\Sigma(\underline{x})$ is the unique extension of $\Sigma|_A(\underline{x})$

to M. Hence by Prop 3.6.6 we have that $\Sigma|_A(x)$ is stationary. ∎

Prop 3.6.8

Let T be stable, $A \subseteq \bar{M}$ and $\Sigma \in S(A)$. If Σ is well-definable then Σ is stationary.

Proof:

Let $M \preceq \bar{M}$ with $A \subseteq M$. Let d be the good defining schema for Σ. By Thm 3.6.1 (a) we have $d(M) \in S(M)$. $d(M)$ is an extension of Σ and is definable over A by d. Hence by prop 3.6.7 Σ is stationary. ∎

Prop 3.6.9

Let T be stable, $A \subseteq \bar{M}$ and d_1, d_2 good defining schemas over A for n-types. If $d_1(A) = d_2(A)$ then $d_1(B) = d_2(B)$ for all $B \supseteq A$.

Proof:

Put $\Sigma = d_1(A) = d_2(A)$. By Prop 3.6.8 Σ is stationary. Let $B \supseteq A$. By corollary 3.6.6 both $d_1(B)$ and $d_2(B)$ are non-forking extensions of Σ to B. Hence $d_1(B) = d_2(B)$. ∎

We now wish to prove converses to propositions 3.6.5–8. First we need some lemmas.

Lemma 3.6.10

Let $A \subseteq M \preceq \bar{M}$ where M is $|A|^+$-saturated. Let $\varphi(\underline{x}, \underline{b})$ be a formula with $\underline{b}$ a tuple in M. Suppose that for all tuples $\underline{c}_1, \underline{c}_2$ in M, if $tp(\underline{c}_1/A) = tp(\underline{c}_2/A)$ and $\models \varphi(\underline{c}_1, \underline{b})$,

then also $\models \varphi(\underline{c}_2, \underline{b})$. Then there is a $\psi(\underline{x}, \underline{a}) \in L(A)$ such that $M \models (\forall \underline{x})(\psi(\underline{x}, \underline{a}) \Leftrightarrow \varphi(\underline{x}, \underline{b}))$.

<u>Lemma 3.6.11</u>

Let T be stable, $A \subseteq M \leq \overline{M}$, $\Sigma(\underline{x}) \in S(A)$ and $\varphi(\underline{x}, \underline{y}) \in L$. Then there are defining schemas $d_1, \ldots, d_k$ $(k < \omega)$ for types over M which are non-forking extensions of $\Sigma(\underline{x})$ such that, if d is a defining schema for a non-forking extension of $\Sigma(\underline{x})$ over M then for some i such that $1 \leq i \leq k$ we have $\models (\forall \underline{y})(d\varphi(\underline{y}) \Leftrightarrow d_i\varphi(\underline{y}))$.

The proofs of the above two lemmas are straightforward and so are omitted. (Notice that Lemma 3.6.10 does not require T to be stable.) The next lemma is the main one needed to prove the results we wish to obtain and is non-trivial. We therefore give a proof.

<u>Lemma 3.6.12</u> (Poizat [77])

Let T be stable, $A \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S_n(A)$. Then $\Sigma(\underline{x})$ has a defining schema d such that for any $B \supseteq A$, $\varphi(\underline{x}, \underline{y}) \in L$ and tuple $\underline{b}$ in B we have $\models d\varphi[\underline{b}]$ iff $\varphi(\underline{x}, \underline{b})$ is in all non-forking extensions of $\Sigma(\underline{x})$ in $S_n(B)$.

<u>Proof:</u>

Let $M \leq \overline{M}$ with $A \subseteq M$, M $|A|^+$-saturated and homogeneous in sufficiently large cardinality so that if $\underline{c}_1, \underline{c}_2$ a tuples in M with $tp(\underline{c}_1/A) = tp(\underline{c}_2/A)$ then the identity map on A extends to an automorphism of M mapping $\underline{c}_1$ onto $\underline{c}_2$. Let $\varphi(\underline{x}, \underline{y}) \in L$. Let $d_1, \ldots, d_k$ be defining schemas given by Lemma 3.6.11. For all $i = 1, \ldots, k$ let $\sigma_i(\underline{y})$ be the formula $d_i\varphi(\underline{y})$. Note that for all $i = 1, \ldots, k$ $\sigma_i(\underline{y}) \in L(M)$. We then have for any $\underline{b}$ in M, $\models \bigwedge_{i=1}^{k} \sigma_i[\underline{b}]$ iff $\varphi(\underline{x}, \underline{b}) \in \Delta(\underline{x})$ for all non-forking extensions

$\Delta(\underline{x})$ of $\Sigma(\underline{x})$ to M. Now put $\sigma(\underline{y}) = \bigwedge_{i=1}^{k} \sigma_i(\underline{y})$. Then $\sigma(\underline{y})$ is in $L(M)$. Suppose that $\varphi(\underline{x}, \underline{b})$ is in all non-forking extensions of $\Sigma(\underline{x})$ to M. Let $\underline{b}'$ be a tuple in M with $tp(\underline{b}/A) = tp(\underline{b}'/A)$. Then the identity map on A extends to an automorphism of M mapping $\underline{b}$ onto $\underline{b}'$. Hence $\varphi(\underline{x}, \underline{b}')$ is also in all non-forking extensions of $\Sigma(\underline{x})$ to M. We thus have: $tp(\underline{b}/A) = tp(\underline{b}'/A)$ implies, $\models \sigma[\underline{b}]$ iff $\models \sigma[\underline{b}']$, for all $\underline{b}, \underline{b}'$ in M. By lemma 3.6.10 there is some formula $\psi(\underline{y})$ in $L(A)$ such that $\models (\forall \underline{y})(\psi(\underline{y}) \Leftrightarrow \sigma(\underline{y}))$. Let d be the defining schema for $\Sigma(\underline{x})$ given by $d\varphi(\underline{y})$ is the $\psi(\underline{y})$ obtained above for each $\varphi(\underline{x}, \underline{y}) \in L$. Then we have for all tuples $\underline{b}$ in M:

$\models d\varphi[\underline{b}]$ iff $\varphi(\underline{x}, \underline{b})$ is in all non-forking extensions of $\Sigma(\underline{x})$ to M. (*)

By (*) we have that the formulae $\neg(d\varphi(\underline{y}) \Leftrightarrow \varphi(\underline{x}, \underline{y}))$ is not represented in any non-forking extension of $\Sigma(\underline{x})$ to M. Thus for any $N \preceq \overline{M}$ with $A \subseteq N$ and $\Delta(\underline{x}) \in S_n(N)$ a non-forking extension of $\Sigma(\underline{x})$, $\neg(d\varphi(\underline{y}) \Leftrightarrow \varphi(\underline{x}, \underline{y}))$ is not represented in $\Delta(\underline{x})$. Hence (*) holds if we replace M by an arbitrary model N containing A and hence also if we replace M by any set containing A. ∎

We immediately have the following corollary:

<u>Corollary 3.6.13</u>

Let T be stable, $A \subseteq M$ and $\Sigma(\underline{x}) \in S(A)$. Then:

a) $\Sigma(\underline{x})$ has a defining schema d such that for all $B \supseteq A$ $d(B)$ is consistent with T.

b.) $\Sigma(\underline{x})$ has a defining schema d' such that for all $B \supseteq A$, $\varphi(\underline{x}, \underline{y}) \in L$ and $\underline{b}$ in B, $\models d'\varphi[\underline{b}]$ iff $\Sigma(\underline{x})$ has some non-forking extension $\Delta(\underline{x}) \in S_n(B)$ with $\varphi(\underline{x}, \underline{b}) \in \Delta(\underline{x})$.

Proof:

For (a.) let d be as in lemma 3.6.12", for (b.) let $d'\varphi(\underline{y})$ be $\neg d(\neg\varphi)(\underline{y})$ for all $\varphi(\underline{x},\underline{y}) \in L$.

∎

We can now prove the desired converses to prop. 3.6.5–8.

Prop 3.6.14

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma(\underline{x}) \in S(M)$. If $\Sigma(\underline{x})$ does not fork over A then $\Sigma(\underline{x})$ is definable almost over A.

Proof:

Let $\varphi(\underline{x},\underline{y}) \in L$. Let d be a defining schema for $\Sigma|_A(\underline{x})$ as given by lemma 3.6.12. Let $\sigma(\underline{x},\underline{y}_1,\underline{y}_2)$ be the formula $\varphi(\underline{x},\underline{y}_1) \Leftrightarrow \varphi(\underline{x},\underline{y}_2)$ and let $E_\varphi(\underline{y}_1,\underline{y}_2)$ be $d\sigma(\underline{y}_1,\underline{y}_2)$. Now $\Sigma(\underline{x})$ is a non-forking extension of $\Sigma|_A(\underline{x})$ so we have that $E_\varphi(\underline{b}_1,\underline{b}_2)$ implies $\varphi(\underline{x},\underline{b}_1) \Leftrightarrow \varphi(\underline{x},\underline{b}_2) \in \Sigma(\underline{x})$ for all $\underline{b}_1, \underline{b}_2$ in M. For all $\underline{b}_1, \underline{b}_2$ in M, $\models E_\varphi(\underline{b}_1,\underline{b}_2)$ iff $\varphi(\underline{x},\underline{b}_1) \Leftrightarrow \varphi(\underline{x},\underline{b}_2)$ is in every non-forking extension of $\Sigma|_A(\underline{x})$ to M. Since M is a model we clearly have that E_φ is a reflexive and symmetric relation. For transitivity let $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ be tuples in M such that both $\models E_\varphi(\underline{b}_1,\underline{b}_2)$, $\models E_\varphi(\underline{b}_2,\underline{b}_3)$. Let $\Delta(\underline{x})$ be any non-forking extension of $\Sigma|_A(\underline{x})$. Thus $\varphi(\underline{x},\underline{b}_1) \Leftrightarrow \varphi(\underline{x},\underline{b}_2) \in \Delta(\underline{x})$ and $\varphi(\underline{x},\underline{b}_2) \Leftrightarrow \varphi(\underline{x},\underline{b}_3) \in \Delta(\underline{x})$. Hence $\varphi(\underline{x},\underline{b}_1) \Leftrightarrow \varphi(\underline{x},\underline{b}_3) \in \Delta(\underline{x})$ and so $\models E_\varphi(\underline{b}_1,\underline{b}_3)$. So E_φ is an equivalence relation. Suppose E_φ has infinitely many blocks, we obtain a contradiction. Let $\{\underline{b}_i : i<\omega\}$ be a set of pairwise non-equivalent tuples in M (w.r.t E_φ). For each i,j such that $i<j<\omega$ we have some $\Delta_{i,j}(\underline{x})$ a non-forking extension of $\Sigma|_A(\underline{x})$ over M such that $\neg\sigma(\underline{x},\underline{b}_i,\underline{b}_j) \in \Delta_{i,j}(\underline{x})$. But there are only finitely many restrictions to $\varphi(\underline{x},\underline{y})$ of non-forking extensions of $\Sigma|_A(\underline{x})$ to M. ($\mathrm{mult}_\varphi(\Sigma|_A)$ is finite.) Hence there are $i<j<k<\omega$ and Δ a non-forking extension of $\Sigma|_A$ to M such that $\neg\sigma(\underline{x},\underline{b}_i,\underline{b}_j)$,

$\neg\sigma(\underline{x}, \underline{b}_j, \underline{b}_k)$, $\neg\sigma(\underline{x}, \underline{b}_i, \underline{b}_k) \in \Delta(\underline{x})$. This is a contradiction and so E_p has only finitely many blocks. Hence $E_p \in FE(A)$ and the proof is complete.

■

Prop. 3.6.15

Let T be stable, $A \subseteq B \subseteq \overline{M}$ and $\Sigma \in S(B)$. If Σ does not fork over A then Σ is definable almost over A.

Proof:

Let $M \preceq \overline{M}$ with $B \subseteq M$ and let Δ be a non-forking extension of Σ to M. Then Δ does not fork over A and so by Prop. 3.6.14. Δ is definable almost over A. Hence $\Sigma = \Delta|_B$ is definable almost over A (by the same weak defining schema.)

■

Prop 3.6.16

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma(\underline{x}) \in S(M)$. If $\Sigma(\underline{x})$ does not fork over A and $\Sigma|_A$ is stationary, then $\Sigma(\underline{x})$ is definable over A.

Proof:

Let d be a defining schema for $\Sigma|_A$ as given by Lemma 3.6.12. Then we have for each $\varphi(\underline{x}, \underline{y}) \in L$ and tuple $\underline{b}$ in M, $\models d\varphi[\underline{b}]$ iff $\varphi(\underline{x}, \underline{b})$ is in every non-forking extension of $\Sigma|_A$ to M, iff $\varphi(\underline{x}, \underline{b}) \in \Sigma(\underline{x})$ (since Σ is the only non-forking extension of $\Sigma|_A$ to M by stationarity.) Hence d is a defining schema for $\Sigma(\underline{x})$ over A.

■

From the above results we immediately obtain:

Corollary 3.6.17

Let T be stable, $A \subseteq \overline{M}$ and $\Sigma \in S(A)$. Σ is stationary iff Σ is well-definable. Also d is a good defining schema for Σ, then for all $B \supseteq A$ $d(B)$ is the unique non-forking extension of Σ to B.

<u>Corollary 3.6.18</u>

Let T be stable, $A \subseteq M \preceq \overline{M}$ and $\Sigma \in S(M)$. Then Σ does not fork over A iff Σ is definable over A. ∎

7. The Open Map Theorem

Let T be stable and consider $A \subseteq B \subseteq \overline{M}$. For $n < \omega$ put $X'_n(A,B) = S_n(B) \setminus X_n(A,B)$. Define a map $f : \underline{X'_n(A,B)} \to \underline{S_n(A)}$ by: for all $\Sigma \in X'_n(A,B)$ $f(\Sigma) = \Sigma|A$.

We will show that f is an open map.

<u>Lemma 3.7.1</u> (Lascar and Poizat [79], see also Lascar [76].)

Let T, A, B be as above. If $\varphi(\underline{x}, \underline{b}) \in L(B)$ with $\underline{x}$ an n-tuple of variables and $\underline{b}$ the tuple of all constants from B in φ. Then there is a $\psi(\underline{x}) \in L(A)$ such that for all $\Sigma(\underline{x}) \in S_n(A)$, $\psi(\underline{x}) \in \Sigma(\underline{x})$ iff there is a $\Delta(\underline{x}) \in X'_n(A,B)$ such that $\varphi(\underline{x}, \underline{b}) \in \Delta(\underline{x})$ and $\Sigma = f(\Delta)$.

<u>Proof:</u>

Let $\varphi(\underline{x}, \underline{b})$ be as above. Let $\Gamma(\underline{y}) = tp(\underline{b}/A)$. By corollary 3.6.13 (b.) there is a $\psi(\underline{x}) \in L(A)$ such that for all tuples $\underline{c}$:

$\models \psi[\underline{c}]$ iff $\Gamma(\underline{y})$ has a non-forking extension over $A \cup \underline{c}$, say $\Gamma'(\underline{y})$ with $\varphi(\underline{c}, \underline{y}) \in \Gamma'(\underline{y})$ (*)

Suppose $\Sigma(\underline{x}) \in S_n(A)$ and $\Sigma(\underline{x})$ has a non-forking extension $\Delta(\underline{x})$ in $S_n(B)$ with $\varphi(\underline{x}, \underline{b}) \in \Delta(\underline{x})$. Let $\underline{c}$ realize $\Delta(\underline{x})$. Hence $tp(\underline{c}/B)$ does not fork over A and so neither does $tp(\underline{c}/(A \cup \underline{b}))$.

By forking symmetry $tp(\underline{b}/(A \cup \underline{c}))$ does not fork over A. But $\models \varphi[\underline{c}, \underline{b}]$ and $tp(\underline{b}/(A \cup \underline{c}))$ is a non-forking extension of $\Gamma(\underline{y})$ and so by $(*)$ we have $\models \psi[\underline{c}]$. Now $\psi(\underline{x}) \in L(A)$ and $tp(\underline{c}/A) = \Sigma(\underline{x})$, hence $\psi(\underline{x}) \in \Sigma(\underline{x})$.

Conversely if $\Sigma(\underline{x}) \in S_n(A)$ with $\psi(\underline{x}) \in \Sigma(\underline{x})$ let $\underline{c}$ realize $\Sigma(\underline{x})$. Then $\models \psi[\underline{c}]$ and so by $(*)$ $\Gamma(\underline{y})$ has a non-forking extension $\Gamma'(\underline{y})$ over $A \cup \underline{c}$ such that $\varphi(\underline{c}, \underline{y}) \in \Gamma'(\underline{y})$. Let $\underline{b}'$ realize $\Gamma'(\underline{y})$. Then $\models \varphi[\underline{c}, \underline{b}]$. Now $tp(\underline{b}'/A) = tp(\underline{b}/A) = \Gamma(\underline{y})$. Hence the identity map on A extends to an automorphism of $\overline{M}$ mapping $\underline{b}'$ onto $\underline{b}$. Let $\underline{c}'$ be the image of $\underline{c}$ under this automorphism. Then we have:

(a.) $\models \varphi[\underline{c}', \underline{b}']$

(b.) $tp(\underline{b}/(A \cup \underline{c}'))$ does not fork over A.

(c.) $tp(\underline{c}'/A) = \Sigma(\underline{x})$

By (b.) and forking symmetry $tp(\underline{c}'/(A \cup \underline{b}))$ does not fork over A. Let $\Delta(\underline{x}) \in S_n(B)$ be a non-forking extension of $tp(\underline{c}'/(A \cup \underline{b}))$. Then $\Delta(\underline{x})$ is a non-forking extension of $\Sigma(\underline{x})$. By (a.) $\varphi(\underline{x}, \underline{b}) \in \Delta(\underline{x})$. ∎

<u>Thm 3.7.2</u> (open map Thm.)

Let T, A, B be as in lemma 3.7.1. Then the map $f: \underline{X_n'(A, B)} \to \underline{S_n(A)}$ given by $f(\Sigma) = \Sigma|_A$, is open.

<u>Proof:</u>

This follows immediately from lemma 3.7.1 and Prop 3.3.1. ∎

<u>Corollary 3.7.3</u>

Let T be stable, $A \subseteq B \subseteq \overline{M}$, $\Sigma \in S_n(A)$ and $\Delta \in S_n(B)$ a non-forking extension of Σ. Then if Δ is isolated in $\underline{S_n(B)}$, Σ is isolated in $\underline{S_n(A)}$.

Proof:

Immediate from Thm 3.7.2. ∎

Notice that the above corollary is interesting since isolated points in $\underline{S_n(A)}$ and $\underline{S_n(B)}$ are principal types. (Prop 3.3.2).

8. The Finite Equivalence Relation Theorem:

In this section we prove an important result concerning finite equivalence relations.

First some lemmas:

Lemma 3.8.1

Let T be stable, $A \subseteq B \subseteq \overline{M}$. Let $\underline{a}$, $\underline{b}$ be tuples and let $\underline{c}$ be the tuple obtained by joining $\underline{a}$ and $\underline{b}$. If $tp(\underline{c}/B)$ does not fork over A then $tp(\underline{b}/(B \cup \underline{a}))$ does not fork over $A \cup \underline{a}$.

Proof:

Suppose $tp(\underline{c}/B)$ does not fork over A. By forking symmetry $tp(B/(A \cup \underline{c} \cup \underline{b}))$ does not fork over A. Hence by Thm 3.1.3(c) $tp(B/(A \cup \underline{a} \cup \underline{b}))$ does not fork over $A \cup \underline{a}$. By forking symmetry $tp(\underline{b}/(B \cup \underline{a}))$ does not fork over $A \cup \underline{a}$.

∎

Lemma 3.8.2

Let T, A, B, $\underline{a}$, $\underline{b}$, $\underline{c}$ be as in lemma 3.8.1. If $tp(\underline{a}/B)$ and $tp(\underline{b}/(B \cup \underline{a}))$ do not fork over A and $A \cup \underline{a}$ respectively then $tp(\underline{c}/B)$ does not fork over A.

(The proof of this is similar to that of lemma 3.8.1, using forking symmetry and Thm 3.1.3(c).)

We can now prove the last main result of this chapter.

<u>Thm 3.8.3</u> (The Finite Equivalence Relation Thm.)
Let T be stable, $A \subseteq M \preceq \bar{M}$ and $\Sigma_1(\underline{x}), \Sigma_2(\underline{x}) \in S_n(M)$ with $\Sigma_1(\underline{x}) \neq \Sigma_2(\underline{x})$. Suppose both $\Sigma_1(\underline{x})$ and $\Sigma_2(\underline{x})$ do not fork over A. Then there is an $E(\underline{x}, \underline{y}) \in FE^n(A)$ such that:
$\Sigma_1(\underline{x}) \cup \Sigma_2(\underline{y}) \vdash \neg E(\underline{x}, \underline{y})$.
(Where obviously we take into account the difference between $\Sigma_2(\underline{x})$ and $\Sigma_2(\underline{y})$.)

<u>Proof:</u>
We need to find realizations $\underline{a}_1, \underline{a}_2$ of $\Sigma_1(\underline{x}), \Sigma_2(\underline{x})$ with $\models \neg E(\underline{a}_1, \underline{a}_2)$ for some $E \in FE^n(A)$. So let $\underline{a}_1, \underline{a}_2$ be realizations of $\Sigma_1(\underline{x}), \Sigma_2(\underline{x})$ such that $tp(\underline{a}_2/(M \cup \underline{a}))$ does not fork over M and hence not over A. Since $\Sigma_1(\underline{x}) \neq \Sigma_2(\underline{x})$ there is a tuple $\underline{d}$ in M and $\varphi(\underline{x}, \underline{z}) \in L$ such that $\models \varphi[\underline{a}_1, \underline{d}]$ and $\models \neg\varphi[\underline{a}_2, \underline{d}]$. By Lemma 3.8.2 $tp(\underline{c}/M)$ does not fork over A and so neither does $tp(\underline{c}/(A \cup \underline{d}))$ where $\underline{c}$ is the tuple obtained by joining $\underline{a}_1$ and $\underline{a}_2$. By forking symmetry $tp(\underline{d}/(A \cup \underline{a}_1 \cup \underline{a}_2))$ does not fork over A. Let $N \preceq \bar{M}$ with $A \cup \underline{a}_1 \cup \underline{a}_2 \subseteq N$. Let Δ be a non-forking extension of $tp(\underline{d}/(A \cup \underline{a}_1 \cup \underline{a}_2))$ over N. Hence Δ does not fork over A. By prop. 3.6.14 Δ is definable almost over A. Then the weak defining schema for Δ is also a weak defining schema for $tp(\underline{d}/(A \cup \underline{a}_1 \cup \underline{a}_2))$ and so this type is also definable almost over A. Let $E \in FE(A)$ be such that for all $\underline{c}_1, \underline{c}_2$ in $A \cup \underline{a}_1 \cup \underline{a}_2$, $\models E(\underline{c}_1, \underline{c}_2)$ implies, $\models \varphi[\underline{c}_1, \underline{d}]$ iff $\models \varphi[\underline{c}_2, \underline{d}]$. Since $\models \varphi[\underline{a}_1, \underline{d}]$ and $\models \neg\varphi[\underline{a}_2, \underline{d}]$ we get $\models \neg E(\underline{a}_1, \underline{a}_2)$ as desired. Now if $\underline{b}_1, \underline{b}_2$ are arbitrary realizations of $\Sigma_1(\underline{x})$ and $\Sigma_2(\underline{x})$ there is a tuple $\underline{e}$ in M such that $\models E(\underline{a}_1, \underline{e})$. (Any block of E intersects any model.) Now $tp(\underline{b}_1/M) = tp(\underline{a}_1/M)$ and so $\models E(\underline{a}_1, \underline{b}_1)$. Similary we get $\models E(\underline{a}_2, \underline{b}_2)$ and so $\models \neg E(\underline{b}_1, \underline{b}_2)$.

Hence we have $\Sigma_1(\underline{x}) \cup \Sigma_2(\underline{y}) \vdash \neg E(\underline{b}_1, \underline{b}_2)$.

$\blacksquare$

<u>Corollary 3.8.4</u>

Let T be stable, $A \subseteq B \subseteq \overline{M}$ and $\Sigma_1(\underline{x})$, $\Sigma_2(\underline{x})$ distinct types in $S_n(B)$ both of which do not fork over A. Then there is an $E(\underline{x},\underline{y})$ in $FE^n(A)$ such that: $\Sigma_1(\underline{x}) \cup \Sigma_2(\underline{y}) \vdash \neg E(\underline{x}, \underline{y})$.

<u>Proof:</u>

Suppose for the sake of contradiction the result is false. Then since $FE^n(A)$ is closed under finite conjunctions it follows that the set $\Gamma(\underline{x}, \underline{y}) = \Sigma_1(\underline{x}) \cup \Sigma_2(\underline{y}) \cup \{E(\underline{x},\underline{y}) : E \in FE^n(A)\}$ is consistent. Let $\Delta(\underline{x}, \underline{y}) \in S(B)$ be an extension of $\Gamma(\underline{x}, \underline{y})$. Let $M \preceq \overline{M}$ with $B \subseteq M$ and let $\Delta'(\underline{x},\underline{y})$ be a non-forking extension of $\Delta(\underline{x},\underline{y})$ to M. Let $\underline{a}_1$, $\underline{a}_2$ be tuples such that $\underline{c}$, the tuple obtained by joining them, realizes $\Delta'(\underline{x},\underline{y})$ (with $\underline{a}_1$ substituted into $\underline{x}$ and $\underline{a}_2$ into $\underline{y}$) Then we have that $tp(\underline{a}_i/M)$ does not fork over A for $i=1,2$. Now $tp(\underline{a}_1/B) = \Sigma_1 \neq \Sigma_2 = tp(\underline{a}_2/B)$ and so $tp(\underline{a}_1/M) \neq tp(\underline{a}_2/M)$. By Thm 3.8.3. there is $E \in FE^n(A)$ such that $\models \neg E(\underline{a}_1, \underline{a}_2)$. This is a contradiction since $\Gamma(\underline{x},\underline{y}) \subseteq tp(\underline{c}/B)$.

$\blacksquare$

4.

ω-Stability and Categoricity

Recall that if T is ω-stable then T is in fact stable in all powers $\beta \geq \aleph_0$. In this chapter we will derive an important result about categoricity by using the above result together with other results about ω-stability. Before doing this we investigate the concept of superstability and examine some characterizations of ω-stability.

1. Superstability and ω-stability.

We first define the concept of superstability.

Definition:

We say that T is <u>superstable</u> iff T is α-stable for all cardinals $\alpha \geq 2^{\aleph_0}$. (Equivalently T is superstable iff for all $n<\omega$ and $M \leq \overline{M}$ with $|M| \geq 2^{\aleph_0}$ we have $|S_n(M)| = |M|$.)

We will use the concept of superstability to characterize ω-stability. Before doing this we prove a result which gives us some equivalences of superstability.

<u>Lemma 4.1.1</u> (Lascar and Poizat [79])

Let $\Sigma \in S_n(M)$ and $M \prec N$ where N is a $|M|^+$-saturated model of T. Let $\Gamma \in O_n(T)$ with $cl(\Sigma) \subseteq \Gamma$. Then there is a $\Delta \in S_n(M)$ such that $\Sigma \subseteq \Delta$ and $cl(\Delta) = \Gamma$.

<u>Lemma 4.1.2</u> (Lascar and Poizat [79])

Let $M, N \leq \overline{M}$, $\alpha > |M|$, N α-saturated and $M \prec N$. Let $\Sigma \in S_n(M)$ and let $\Gamma \in O_n(T)$ be such that $cl(\Sigma) \subset \Gamma$. Then there are distinct $\Delta_i \in S_n(N)$ for all $i<\alpha$ such that $\Sigma \subseteq \Delta_i$ and $cl(\Delta_i) = \Gamma$

for all $i<\alpha$.

(Lemma 4.1.1 follows from a straightforward compactness argument. Lemma 4.1.2 can be proved using induction and Lemma 4.1.1.)

If the fundamental order $O_n(T)$ satisfies the ascending chain condition we say that $O_n(T)$ is <u>well-founded</u>.

<u>Lemma 4.1.3</u> (Lascar and Poizat [79])

Suppose $O_n(T)$ is not well-founded. Let $\alpha \geq \aleph_0$. Then there is $M \leq \bar{M}$ with $|M| = \alpha$ and $|S_n(M)| \geq \alpha^{\aleph_0}$.

<u>Proof:</u>

Let $\{\Gamma_i : i<\omega\} \subseteq O_n(T)$ with $\Gamma_i \subset \Gamma_{i+1}$ for all $i<\omega$. We define inductively, models M_k of T and types $\Sigma_f \in S_n(M_k)$ for all $f \in {}^k\alpha$, $k<\omega$, such that:

(a) $|M_k| = \alpha$ for all $k<\omega$

(b) $M_k \prec M_{k+1}$ for all $k<\omega$

(c) $f, g \in {}^{\omega>}\alpha$ and $f \subseteq g$ implies $\Sigma_f \subseteq \Sigma_g$.

(d) If $f \in {}^{\omega>}\alpha$ and $i<j<\omega$, then $\Sigma_{f+i} \neq \Sigma_{f+j}$

(e) If $f \in {}^k\alpha$ then $cl(\Sigma_f) = \Gamma_k$

(where ${}^{\omega>}\alpha$ denotes $\bigcup_{k<\omega} {}^k\alpha$ and for $f \in {}^k\alpha$ and $i<\omega$ $f+i$ denotes the extension of f to $k+1$ with $f(k) = i$.)

Suppose that we have found M_k and $\Sigma_f \in S_n(M_k)$ for all $f \in {}^k\alpha$. By Lemma 4.1.2, for each $f \in {}^k\alpha$ there is a model $M_f \succ M_k$ and distinct types $\Delta_{f,i} \in S_n(M_f)$ for all $i<\alpha$, such that $\Sigma_f \subseteq \Delta_{f,i}$ and $cl(\Delta_{f,i}) = \Gamma_{k+1}$ for all $i<\alpha$. We may choose M_f so that $|M_f| = \alpha$ for all $f \in {}^k\alpha$. Now let M_{k+1} be any model of such that $|M_{k+1}| = \alpha$ and $M_f \subseteq M_{k+1}$ for all $f \in {}^k\alpha$. Let $\Sigma_{f+i} \in S_n(M_{k+1})$ be an heir of $\Delta_{f,i}$ for all $f \in {}^k\alpha$ and $i<\alpha$. Then conditions (a)–(e) are satisfied and the induction is complete. Now let $M = \bigcup_{k<\omega} M_k$. Then $|M| = \alpha$.

For $f \in {}^{\omega}\alpha$ let $\Sigma_f \in S_n(M)$ be $\bigcup_{k<\omega} \Sigma_{f|k}$. Then if $f, g \in {}^{\omega}\alpha$ with $f \neq g$ we have $\Sigma_f \neq \Sigma_g$ and the result is proved. ∎

We can now prove an interesting result linking well-boundedness, forking and superstability.

<u>Prop 4.1.4</u> (Lascar and Poizat [79] extending Shelahs result in Shelah [78])
Let T be stable. The following are equivalent for any $n<\omega$.

(a) $O_n(T)$ is well-founded

(b) There is no chain $\{(A_k, \Sigma_k)\}_{k<\omega}$ with $A_k \subseteq A_{k+1} \subseteq \bar{M}$ and $\Sigma_{k+1} \in S_n(A_{k+1})$ a forking extension of $\Sigma_k \in S_n(A_k)$ for all $k<\omega$.

(c) For any $B \subseteq \bar{M}$ and $\Sigma \in S_n(B)$ there is a finite $A \subseteq B$ such that Σ does not fork over A.

(d) T is superstable.

<u>Proof:</u>

(a) $\Rightarrow$ (b):

Assume (a) but suppose there does exist a chain as described in (b). Then $\{\beta(\Sigma_k) : k<\omega\}$ is an infinite strictly increasing chain in $O_n(T)$ contradicting (a).

(b) $\Rightarrow$ (c):

Let $B \subseteq \bar{M}$ and $\Sigma \in S_n(B)$. Suppose (c) is false. Let $A_k \subseteq B$ with A_k finite such that Σ forks over A_k. Hence by Prop 3.1.6 there is a tuple $\underline{b}$ in B such that $\Sigma|_{A_k \cup \underline{b}}$ forks over A_k. Let A_{k+1} be $A_k \cup \underline{b}$. Hence by induction we obtain an increasing sequence of (finite) subsets of B, $\{A_k : k<\omega\}$ such that for all $k<\omega$ $\Sigma|_{A_{k+1}}$ forks over A_k. This contradicts (b).

(c) $\Rightarrow$ (d):

Let $M \leq \bar{M}$ with $|M| = \alpha \geq 2^{\aleph_0}$. Assuming (c.) we have for each $\Sigma \in S_n(M)$ a finite $A_\Sigma \subseteq M$ such that Σ does not fork over A_Σ. There are at most α distinct A_Σ's for $\Sigma \in S_n(M)$. For each A_Σ we have $|S_n(A_\Sigma)| \leq 2^{\aleph_0}$ since A_Σ is finite. Also for each $\Delta \in S_n(A_\Sigma)$, Δ has at most $2^{\aleph_0}$ non-forking extensions over M. (By Thm 3.4.2) Thus we have $|S_n(M)| \leq 2^{\aleph_0} \cdot 2^{\aleph_0} \alpha = 2^{\aleph_0} \cdot \alpha$. Hence T is α-stable. Since $M \leq \bar{M}$ was arbitrary with $|M| = \alpha > 2^{\aleph_0}$ we have that T is superstable.

(d.) $\Rightarrow$ (a.):

Assume (d.) but suppose $O_n(T)$ is not well-founded. Let $\alpha \geq 2^{\aleph_0}$ with $\alpha < \alpha^{\aleph_0}$. By Lemma 4.1.3 there is a model $M \leq \bar{M}$ with $|M| = \alpha$ and $|S_n(M)| \geq \alpha^{\aleph_0}$. Hence T is not α-stable contradicting superstability.

■

We have the following corollary:

<u>Corollary 4.1.5.</u> (Pillay [83])

Let T be stable. Then either:

a) T is superstable, or

b.) For any $\alpha \geq \aleph_0$, T is α-stable iff $\alpha = \alpha^{\aleph_0}$

<u>Proof:</u>

Suppose T is not superstable. Let $\alpha \geq \aleph_0$ with $\alpha = \alpha^{\aleph_0}$ and let $M \leq \bar{M}$ with $|M| = \alpha$. There are at most $\alpha^{\aleph_0}$ defining schemas over M and hence (since T is stable) at most $\alpha^{\aleph_0}$ types over M. Hence $|S(M)| = \alpha^{\aleph_0} = \alpha$ and so T is α-stable. If $\alpha \geq \aleph_0$ with $\alpha < \alpha^{\aleph_0}$ then by Prop 4.1.4 $O_1(T)$ is not well-founded and so by lemma 4.1.3 there is $M \leq \bar{M}$ with $|M| = \alpha$ but $|S_1(M)| \geq \alpha^{\aleph_0}$. Hence T is not α-stable.

■

Before we can prove the main results of this section we need the following lemma which was mentioned in the section on multiplicity in Chapter 3.

<u>Lemma 4.1.6</u> (Shelah [78])

Let T be stable, $A \subseteq \overline{M}$ and $\Sigma(\underline{x}) \in S_n(A)$. Then $\mathrm{mult}(\Sigma)$ is either finite or $2^{\aleph_0}$.

<u>Proof:</u>

We already know that $\mathrm{mult}(\Sigma) \leq 2^{\aleph_0}$ and so it suffices to show that if $\mathrm{mult}(\Sigma) \geq \aleph_0$ then $\mathrm{mult}(\Sigma) \geq 2^{\aleph_0}$. Let $\mathrm{mult}(\Sigma) \geq \aleph_0$. Using the Finite Equivalence Relation Thm (Thm 3.8.3) we can show that for each $E \in FE^n(A)$ there is an $E' \in FE^n(A)$ such that $\Sigma(\underline{x}) \cup \Sigma(\underline{y}) \cup \{E(\underline{x},\underline{y}), \neg E'(\underline{x},\underline{y})\}$ is consistent. Hence there are $E_i \in FE^n(A)$ for $i<\omega$ such that for all $i<\omega$:

$$\Sigma(\underline{x}) \cup \Sigma(\underline{y}) \cup \{E_i(\underline{x},\underline{y}), \neg E_{i+1}(\underline{x},\underline{y})\} \text{ is consistent.} \qquad (*)$$

By replacing $E_i(\underline{x},\underline{y})$ by $\bigwedge_{j\leq i} E_j(\underline{x},\underline{y})$ for all $i<\omega$ we can assume that:

$$\vdash E_i(\underline{x},\underline{y}) \Rightarrow E_j(\underline{x},\underline{y}) \quad \text{for all } j \leq i < \omega \qquad (**)$$

<u>Claim 1:</u> If $\underline{a}$ is a tuple realizing $\Sigma(\underline{x})$ and $i<\omega$, then $\Sigma(\underline{y}) \cup \{E_i(\underline{a},\underline{y}), \neg E_{i+1}(\underline{a},\underline{y})\}$ is consistent with $\overline{M}$.

Proof: By $(*)$ there is a $\underline{b}$ realizing Σ such that $\Sigma(\underline{y}) \cup \{E_i(\underline{b},\underline{y}), \neg E_{i+1}(\underline{b},\underline{y})\}$ is consistent. But $tp(\underline{a}/A) = tp(\underline{b}/A)$ and so $\Sigma(\underline{y}) \cup \{E_i(\underline{a},\underline{y}), \neg E_{i+1}(\underline{a},\underline{y})\}$ is consistent.

<u>Claim 2:</u> Let $k<\omega$ and suppose $\underline{a}$ is a tuple such that the set $\Sigma(\underline{x}) \cup \{E_k(\underline{x},\underline{a})\}$ is consistent. Then there are tuples $\underline{b}$ and $\underline{c}$ such that $\vdash E_k(\underline{b},\underline{a}) \wedge E_k(\underline{c},\underline{a}) \wedge \neg E_{k+1}(\underline{b},\underline{c})$ and such that both $\Sigma(\underline{x}) \cup \{E_{k+1}(\underline{x},\underline{b})\}$, $\Sigma(\underline{x}) \cup \{E_{k+1}(\underline{x},\underline{c})\}$

are consistent.

Proof: Let $\underline{b}$ realize $\Sigma(\underline{x}) \cup \{E_k(\underline{x}, \underline{a})\}$. By 1. there is a tuple $\underline{c}$ realizing $\Sigma(\underline{x})$ such that $\vdash E_k(\underline{b}, \underline{c}) \wedge \neg E_{k+1}(\underline{b}, \underline{c})$. Then $\models E_k(\underline{b}, \underline{a}) \wedge E_k(\underline{c}, \underline{a})$ and the result follows.

<u>Claim 3:</u> There are n-tuples $\underline{a}_f$ for $f \in {}^{\omega>}2$ such that:

(i) For each $k<\omega$ if $f \in {}^k2$ then:

$$\models E_k(\underline{a}_{f+0}, \underline{a}_f) \wedge E_k(\underline{a}_{f+1}, \underline{a}) \wedge \neg E_{k+1}(\underline{a}_{f+0}, \underline{a}_{f+1})$$

(ii) For each $f \in {}^\omega 2$, $\Sigma_f(\underline{x})$ is consistent, where

$$\Sigma_f(\underline{x}) = \Sigma(\underline{x}) \cup \{E_k(\underline{x}, \underline{a}_{f|k}) : k<\omega\}$$

Proof: By 2. we can define the tuples $\underline{a}_f$, $f \in {}^{\omega>}2$ by induction, so as to satisfy condition (i) and:

(iii) If $f \in {}^k2$ $(k<\omega)$ then $\Sigma(\underline{x}) \cup \{E_k(\underline{x}, \underline{a}_f)\}$ is consistent.

By $(**)$, if $f \in {}^k2$ then $\vdash E_k(\underline{x}, \underline{a}_f) \Rightarrow E_i(\underline{x}, \underline{a}_f)$ for all $i \leq k$. By (i) and $(**)$ we have $\models E_i(\underline{a}_f, \underline{a}_{f|i})$ for all $i \leq k$, and so $\models E_k(\underline{x}, \underline{a}_f) \Rightarrow E_i(\underline{x}, \underline{a}_{f|i})$ for all $i \leq k$. Then by (iii) and compactness we get $\Sigma_f(\underline{x})$ is consistent for each $f \in {}^\omega 2$.

Now by 3(i) we have that if $f, g \in {}^\omega 2$ and $f \neq g$ then the set $\Sigma_f(\underline{x}) \cup \Sigma_g(\underline{x})$ is consistent. Put $B = A \cup \bigcup\{\underline{a}_f : f \in {}^{\omega>}2\}$.

<u>Claim 4:</u> For each $f \in {}^\omega 2$, $\Sigma_f(\underline{x})$ has an extension $\Delta_f(\underline{x})$ in $S_n(B)$ such that $\Delta_f(\underline{x})$ does not fork over A.

Proof: Let $M \preceq \overline{M}$ with $A \subseteq M$ and M, B independent over A. Put

$$\Phi(\underline{x}) = \{\neg\varphi(\underline{x}) : \varphi \in L(M \cup B) \text{ and } \varphi(\underline{x}) \text{ not satisfied in } M\}.$$

We show that $\Sigma_f(\underline{x}) \cup \Phi(\underline{x})$ is consistent. Suppose it is not consistent. Then there are $\varphi_1(\underline{x}), \ldots, \varphi_m(\underline{x}) \in L(M \cup B)$ which are not satisfied in M and a formula $\sigma(\underline{x})$ which is a

conjunction of formulae in $\Sigma_f(\underline{x})$ such that we have:
$\vdash \sigma(\underline{x}) \Rightarrow \bigvee_{i=1}^{m} \varphi_i(\underline{x})$. But $\sigma(\underline{x})$ is satisfied in M and so we get a contradiction. Thus $\Sigma_f(\underline{x}) \cup \Phi(\underline{x})$ is consistent. Let $\Delta'(\underline{x})$ be an extension of $\Sigma_f(\underline{x}) \cup \Phi(\underline{x})$ to a complete type in $S_n(M \cup B)$. Let $\Delta_f(\underline{x}) = \Delta'|_B(\underline{x})$. Then we have $\Sigma_f(\underline{x}) \subseteq \Delta_f(\underline{x})$ and also $\Delta_f(\underline{x})$ does not fork over A. (By Prop. 3.2.4.)

Now the Δ_f's obtained in 4. are pairwise inconsistent and so $\Sigma(\underline{x})$ has at least $2^{\aleph_0}$ non-forking extensions over B. Hence $\text{mult}(\Sigma) \geq 2^{\aleph_0}$ and the result is proved.

∎

We are now able to characterise ω-stability using the concepts of superstability and multiplicity.

<u>Thm 4.1.7.</u> (Pillay [83])

T is ω-stable iff:

(a) T is superstable

(b.) $S(T)$ is countable

(c.) For all $A \subseteq \bar{M}$ and all $\Sigma \in S(A)$, $\text{mult}(\Sigma)$ is finite.

<u>Proof:</u>

$\Rightarrow$: If T is ω-stable then T is α-stable for all $\alpha \geq \aleph_0$, in particular for all $\alpha \geq 2^{\aleph_0}$, and so T is superstable. Also by Prop. 1.1.3 $|S_n(T)| \leq \aleph_0$ for all $n \in \omega$ and so $|S(T)| \leq \aleph_0$. Now suppose $A \subseteq \bar{M}$ and $\Sigma \in S(A)$. By Prop 4.1.4 (c.) there is a finite $B \subseteq A$ such that Σ does not fork over B. Suppose $\text{mult}(\Sigma|_B)$ is infinite. Then by Lemma 4.1.6 $\text{mult}(\Sigma|_B) = 2^{\aleph_0}$. Let $M \preceq \bar{M}$ with $B \subseteq M$ and M countable. Then by Prop. 3.5.2 $\Sigma|_B$ has $2^{\aleph_0}$ non-forking extensions over M and so $|S(M)| = 2^{\aleph_0}$. But this contradicts ω-stability and so $\text{mult}(\Sigma|_B)$ is finite. Since Σ does not fork over B, any non-forking extension of Σ is a non-forking extension

of $\Sigma|_B$. Hence $\mathrm{mult}(\Sigma) \leq \mathrm{mult}(\Sigma|_B)$ and so $\mathrm{mult}(\Sigma)$ is finite.

$\Leftarrow$: Assume (a.), (b.), (c.). Let $M \preceq \bar{M}$ be countable. Let $\Sigma \in S_1(M)$. By superstability and Prop 4.1.4 (c.) there is a finite $A \subseteq M$ such that Σ does not fork over A. By (c.) $\Sigma|_A$ has only finitely many non-forking extensions over M. Now there are only countably many finite subsets $A \subseteq M$ and by (b.) for each such A we get that $S_1(A)$ is countable. Then we have $S_1(M)$ is countable. Hence T is ω-stable. ∎

We can also characterize ω-stability in terms of definability.

<u>Thm 4.1.8</u> (Pillay [83])

T is ω-stable iff:

(a) $S(T)$ is countable.

(b) For all countable $M \preceq \bar{M}$ and all $\Sigma \in S(M)$ there is a finite $A \subseteq M$ such that Σ is definable over A.

<u>Proof</u>

$\Rightarrow$: If T is ω-stable then by Thm 4.1.7 $S(T)$ is countable. Let $M \preceq \bar{M}$ be countable and let $\Sigma \in S(M)$. By Thm 4.1.7 T is superstable and so there is a finite $B \subseteq M$ such that Σ does not fork over B. (Prop 4.1.4 (c.)) By Thm 4.1.7. $\mathrm{mult}(\Sigma|_B)$ is finite. Hence there is a finite A such that $B \subseteq A \subseteq M$ and if Σ_1, Σ_2 are distinct non-forking extensions of $\Sigma|_B$ over M, then $\Sigma_1|_A \neq \Sigma_2|_A$. Let $\Delta = \Sigma|_A$. Then Σ is the only non-forking extension of Δ over M. By Prop. 3.5.2 Δ is stationary and so by Prop 3.6.16. Σ is definable over A.

$\Leftarrow$: Assume (a.), (b.). By (b.) and lemma 1.4.1 all types over all models of T are definable and so by Thm 1.4.2 T is stable. Let

$M \leq \bar{M}$ be countable. By (a.) there are only countably many 1-types over any finite subset of M. By Prop 3.6.9 if $\Sigma, \Delta \in S_1(M)$ are both definable over $A \subseteq M$ and $\Sigma|_A = \Delta|_A$ then $\Delta = \Sigma$. Thus by (b.) we have that $S_1(M)$ is countable and so T is ω-stable.

Superstability and ω-stability are not equivalent in general. For example let L be a language with unary predicate symbols P_n, $n < \omega$. Let T be the theory that says:

$(\exists x)(P_{i_1}(x) \wedge \ldots \wedge P_{i_n}(x) \wedge \neg P_{j_1}(x) \wedge \ldots \wedge \neg P_{j_m}(x))$ for each $\{i_1, \ldots, i_n, j_1, \ldots j_m\} \subseteq \omega$. Then for any $X \subseteq \omega$ the set of formulae $\{P_i(x) : i \in X\} \cup \{\neg P_i(x) : i \notin X\}$ is consistent with T and so can be extended to a complete 1-type of T. Hence $S_1(T)$ is uncountable and so T is <u>not</u> ω-stable. Now if M is a model of T there are $|M|$ types in $S_1(M)$ which are realized in M. If $\Sigma \in S_1(M)$ is not realized in M then Σ is uniquely determined by $X = \{i < \omega : P_i(x) \in \Sigma\}$, and so there are $2^{\aleph_0}$ such types. Hence $|S_1(M)| = \max\{2^{\aleph_0}, |M|\}$. Hence T is α-stable for all $\alpha \geq 2^{\aleph_0}$ ie T is superstable.

A theory may also be stable but not superstable. For example let T be a theory that says that for all $n < \omega$ E_n is an equivalence relation with infinitely many blocks all of which are infinite and for all $n < \omega$ E_{n+1} partitions E_n into infinitely many blocks. Let M be a model of T and let $\Sigma(x) \in S_1(M)$ be a type not realized in M. Then $\Sigma(x)$ is determined by formulae of the forms $E_n(x,a)$ or $\neg E_n(x,a)$ contained in $\Sigma(x)$ where $n < \omega$ and $a \in M$. There are at most $|M|$ different $\Sigma|_{E_n}$ for each $n < \omega$. Hence $|S_1(M)| \leq |M|^{\aleph_0}$ and so T is stable. For each $n < \omega$ there is a $\Sigma_n(x) \in S_1(M)$ such that $E_n(x,a) \in \Sigma_n(x)$ for some $a \in M$ but for all $b \in M$ we have $\neg E_{n+1}(x,b) \in \Sigma_n(x)$ ie $E_{n+1}(x,b) \notin \Sigma_n(x)$. Then for all $n < \omega$ $cl(\Sigma_n) \subset cl(\Sigma_{n+1})$ and so $O_n(T)$ is not well-founded and so by Prop 4.1.4

T is not superstable.

2. Morley's Theorem

Recall that T is said to be α-categorical iff any two models of T of power α are isomorphic.

In this section we will prove Morley's Theorem which states that if T is α-categorical for some $\alpha > \aleph_0$ then T is α-categorical for all $\alpha > \aleph_0$. The proof requires a series of Lemmas. We will give proofs for the lemmas involve ω-stability. For the proofs of the other lemmas we refer the reader to Chang and Keisler [73] pg. 401–413.

Lemma 4.2.1 (Morley [65])

Suppose every model of T of power $\aleph_1$ is saturated. Then every uncountable model of T is saturated. (By Prop. 1.1.1 T will then be α-categorical for all $\alpha \geq \aleph_1$.)

Lemma 4.2.2 (Morley [65])

For all cardinals $\alpha \geq \aleph_0$ there is a model M of T such that $|M| = \alpha$ and for all $B \subseteq M$ $|S_1(B)| \leq |B| + \aleph_0$.

Lemma 4.2.3 (Morley [65])

Let T be α-categorical for some $\alpha > \aleph_0$. Then T is ω-stable.

Proof:

Let $A \subseteq \overline{M}$ with $|A| \leq \aleph_0$. By lemma 4.2.2 there is a $M \preceq \overline{M}$ such that $|M| = \alpha$ and for all $B \subseteq M$ we have $|S_1(B)| \leq |B| + \aleph_0$. Now T is α-categorical and so any $M \preceq \overline{M}$ with $|M| = \alpha$ has this property. Hence we may assume $A \subseteq M$. Then $|S_1(A)| \leq |A| + \aleph_0 = \aleph_0$. Hence T is ω-stable.

The converse of the above lemma does not hold. Let L be a language with unary predicate symbols P_n for $n<\omega$ and constant symbols c_{mn} for $m,n<\omega$. Let T be the theory that says:

a) For each $n<\omega$ P_n is an infinite set

b.) For $n,m<\omega$ with $n\neq m$ P_n and P_m are disjoint.

c.) The constants c_{mn} $(m,n<\omega)$ are all distinct.

d.) For all $n,m<\omega$ $P_n(c_{mn})$

Then T is ω-stable but clearly T is not categorical in any power.

<u>Lemma 4.2.4</u> (Morley [65])

Let T be ω-stable. If $\alpha>\aleph_0$ is regular and $\beta\geq\alpha$, then there is a $M\preceq\overline{M}$ such that $|M|=\beta$ and M is α-saturated.

<u>Proof:</u>

Let $A\preceq\overline{M}$ with $|A|=\beta$. Since T is ω-stable we have that T is β-stable. Hence $|S_1(A)|\leq\beta$. Let $A'\preceq\overline{M}$ be such that $A\preceq A'$ and every type in $S_1(A)$ is realized in A'. Since we have $|S_1(A)|\leq\beta$ we may choose A' so that $|A'|=\beta$. We can thus form an elementary chain $(A_\gamma)_{\gamma<\alpha}$ such that:

(a.) For all $\gamma<\alpha$ $A_\gamma\preceq\overline{M}$ with $|A_\gamma|=\beta$

(b) For all $\gamma<\alpha$ every type in $S_1(A_\gamma)$ is realized in $A_{\gamma+1}$

(c.) If γ is a limit ordinal $A_\gamma=\bigcup_{\delta<\gamma}A_\delta$.

Put $M=\bigcup_{\gamma<\alpha}A_\gamma$. Then since $\beta\geq\alpha$ we have $|M|=\beta$. We have to show that M is α-saturated. Let $X\subseteq B$ with $|X|<\alpha$. Since α is regular there is an ordinal $\gamma<\alpha$ such that $X\subseteq A_\gamma$. Let $\Sigma(x)$ be a type of $Th(M,x)_{x\in X}$. Then $\Sigma(x)$ will be a type in $S_1(X)$ and since $X\subseteq A_\gamma$ and (b.) we have that $\Sigma(x)$ is realized in $A_{\gamma+1}$. Now $A_{\gamma+1}\prec M$ and so $\Sigma(x)$ is realized in M. Hence M is α-saturated.

<u>Lemma 4.2.5</u> (Upward Morley Thm., Morley [65])

If T is $\aleph_1$-categorical then T is α-categorical for all $\alpha \geq \aleph_1$.

<u>Proof:</u>

Suppose T is $\aleph_1$-categorical. By lemma 4.2.3 T is ω-stable. Now $\aleph_1$ is regular and so by lemma 4.2.4 T has a saturated model of power $\aleph_1$. By $\aleph_1$-categoricity every model of T of power $\aleph_1$ is $\aleph_1$-saturated. By lemma 4.2.1 and its remark T is α-categorical for all $\alpha \geq \aleph_1$. ∎

Before proceeding we need the following definitions:

Let $A \subseteq \overline{M}$. Let $\varphi(\underline{x}) \in Fn(A)$. We say that $\varphi(\underline{x})$ is a <u>complete formula</u> of A iff $[\varphi(\underline{x})]$ is an atom in the Boolean algebra $Bn(A)$. If $\varphi(\underline{x})$ is complete and $\sigma(\underline{x}) \in Fn(A)$ with $[\varphi(\underline{x})] \leq [\psi(\underline{x})]$ we say that $\sigma(\underline{x})$ is <u>completable</u> and that $\varphi(\underline{x})$ is an <u>atom of $\sigma(\underline{x})$</u>.

<u>Lemma 4.2.6</u> (Morley [65])

Let T be ω-stable. For all $n < \omega$ and for all $A \subseteq \overline{M}$ every $\sigma(\underline{x})$ in $Fn(A)$ consistent with $Th(\overline{M}, a)_{a \in A}$ is completable.

<u>Proof:</u>

By contradiction. Assume the result is false. Let n be least such that there is $A \subseteq \overline{M}$ and $\sigma(\underline{x}) \in Fn(A)$ consistent with $Th(\overline{M}, a)_{a \in A}$ but incompletable. Suppose $n > 1$. Then if $\underline{x} = (x_1, \ldots, x_n)$ we have that $(\exists x_n)\, \sigma(x_1, \ldots, x_n) \in Fn_{-1}(A)$ and so it has an atom $\varphi(x_1, \ldots, x_{n-1})$. Let $a_1, \ldots, a_{n-1} \in \overline{M}$. Then $Th(\overline{M}, a)_{a \in A} \cup \{\varphi(a_1, \ldots, a_{n-1})\}$ determines a complete theory T' in $L(A \cup \{a_1, \ldots, a_{n-1}\})$. Assume $\sigma(a_1, \ldots, a_{n-1}, x_n)$ has an atom $\theta(a_1, \ldots, a_{n-1}, x_n)$. Then we have that $T' \vdash \theta(a_1, \ldots, a_{n-1}, x_n) \Rightarrow \sigma(a_1, \ldots, a_{n-1}, x_n)$, and so $Th(\overline{M}, a)_{a \in A} \vdash \varphi(x_1, \ldots, x_{n-1}) \wedge$

$\theta(x_1, \ldots, x_n) \Rightarrow \sigma(x_1, \ldots, x_n)$. But $\varphi(x_1, \ldots, x_{n-1}) \wedge \theta(x_1, \ldots, x_n)$ is a complete formula of A which is a contradiction since $\sigma(\underline{x})$ is not completable. Hence $\sigma(a_1, \ldots, a_{n-1}, x_n)$ is not completable. Hence we have shown that we can always reduce the lemma to the case $n=1$. So assume $n=1$, $\sigma(x)$ is consistent with $Th(\bar{M}, a)_{a \in A}$ but that $\sigma(x)$ is incompletable. Now for every $\theta(x)$ consistent with $Th(\bar{M}, a)_{a \in A}$ but incompletable we may choose two consistent formulas $\theta_0(x)$ and $\theta_1(x)$ such that:

a) $Th(\bar{M}, a)_{a \in A} \vdash \theta_0(x) \Rightarrow \theta(x)$

b) $Th(\bar{M}, a)_{a \in A} \vdash \theta_1(x) \Rightarrow \theta(x)$

c) $Th(\bar{M}, a)_{a \in A} \vdash \neg(\theta_0(x) \wedge \theta_1(x))$

Then $\theta_0(x)$ and $\theta_1(x)$ are also incompletable. Starting with $\sigma(x)$ $\sigma(x)$ we use the above process and induction to find formulae $\sigma_f(x)$, $f \in {}^{\omega>}2$ which are consistent with $Th(\bar{M}, a)_{a \in A}$ but not completable. Let B be the set of all constants that occur in the $\sigma_f(x)$'s. Then $|B| \leq \aleph_0$ but by considering the $\sigma_f(x)$'s we see that $|S_1(B)| \geq 2^{\aleph_0}$ contradicting ω-stability. ∎

If M is a model of T and $A \subseteq M$ we say that M is <u>atomic over A</u> iff every n-tuple $\underline{a}$ in M satisfies a complete $\varphi(\underline{x}) \in Fn(A)$. M is called <u>prime</u> iff for every model N of T, $M \prec N$.

<u>Lemma 4.2.7</u> (Morley [65])

Suppose for all $A \subseteq \bar{M}$, every $\sigma(\underline{x}) \in Fn(A)$ consistent with $Th(M, a)_{a \in A}$ is completable. Then for all $A \subseteq \bar{M}$ there is a prime model M of T which is atomic over A.

Combining lemma 4.2.6 and lemma 4.2.7 we get:

<u>Lemma 4.2.8</u>

Let T be ω-stable and $A \subseteq \overline{M}$. Then there is a prime model M of T which is atomic over A. ∎

<u>Lemma 4.2.9</u> (Baldwin and Lachlan [71])

Let T be ω-stable. Let $M \preceq \overline{M}$ with $|M| > \aleph_0$. Then there is a proper elementary extension N of M such that every countable set $\Gamma(x)$ of formulae in $L(M)$, which is realized in N is realized in M.

<u>Proof:</u>

We start by finding a formula $\psi(x) \in L(M)$ such that:

(a.) $\{b \in M : M \models \psi[b]\}$

(b.) For any $\varphi(x) \in L(M)$ at least one of the sets

$\{b \in M : M \models \psi[b] \wedge \varphi[b]\}$, $\{b \in M : M \models \psi[b] \wedge \neg\varphi[b]\}$

is countable.

Suppose there is no such formula. Then for any $\varphi(x) \in L(M)$ satisfied by uncountably many $b \in M$, there are two formulae $\varphi_0(x)$, $\varphi_1(x)$ in $L(M)$ such that:

(i) $\models \varphi_0(x) \Rightarrow \varphi(x)$

(ii) $\models \varphi_1(x) \Rightarrow \varphi(x)$

(iii) $\models \neg(\varphi_0(x) \wedge \varphi_1(x))$

(iv) $\varphi_0(x)$, $\varphi_1(x)$ are satisfied by uncountably many $b \in M$.

Hence by induction we can find formulae $\varphi_f(x)$, $f \in {}^{\omega>}2$ each satisfied by uncountably many $b \in M$. Let $B \subseteq \overline{M}$ be the set of all $b \in M$ occurring in the $\varphi_f(x)$'s. Then we have $|B| \leq |M|$ but $|S_1(B)| > |M|$ and so T is not $|M|$-stable. But this contradicts ω-stability since $|M| > \aleph_0$. Hence the desired $\psi(x) \in L(M)$ exists.

Add a new constant symbol $c \in \overline{M}$ to $L(A)$. Let $\Delta = \{\varphi(c) \in L(M \cup \{c\}) :$ there are only countably many $b \in M$ with $\models \psi[b]$ but $\not\models \varphi[b]\}$.

Then by (a.) and (b.) Δ is a complete theory in $L(M \cup \{c\})$. Also $Th(M, a)_{a \in M} \subseteq \Delta$. By Lemma 4.2.8 Δ has a prime model (N, c)

which is atomic over $M \cup \{c\}$. Now $M \prec N$ but for all $b \in M$, $N \models X[b,c]$ where $X(x,y)$ is $\neg(x=y)$. Hence N is in fact a proper elementary extension of M. Now let $\Gamma(x)$ be a countable set of formulae in $L(M)$ which is realized in M. Let $b \in N$ realize $\Gamma(x)$. Then b satisfies a complete $\Delta \vdash (\exists x)\sigma(x,c)$ and $\Delta \vdash \sigma(x,c) \Rightarrow \gamma(x)$ for all $\gamma(x) \in \Gamma(x)$. $\Gamma(x)$ is countable and so there is a countable $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash (\exists x)\sigma(x,c)$ and $\Delta_0 \vdash \sigma(x,c) \Rightarrow \gamma(x)$ for all $\gamma(x) \in \Gamma(x)$. Now each $\varphi(x)$ in Δ_0 is satisfied by all except countably many $b \in M$ which satisfy $\psi(x)$. Hence there is a $d \in M$ such that (M,d) is a model of Δ_0 with d interpreting c. Then there is a $b' \in M$ such that $M \models \sigma[b',d]$. Hence b' realizes $\Gamma(x)$ in M.

∎

We can now prove Morley's Theorem:

<u>Thm 4.2.10</u> (Morley Categoricity Theorem, Morley [65], proof following Chang and Keisler [73])

If T is α-categorical for some $\alpha > \aleph_0$ then T is α-categorical for all $\alpha > \aleph_0$.

<u>Proof:</u>

Suppose T is α-categorical for $\alpha > \aleph_0$. By lemma 4.2.3 T is ω-stable. Let $M \preceq \bar{M}$ be such that $|M| = \aleph_1$. By lemma 4.2.9 and induction (taking unions at limit ordinals) we obtain an elementary extension N of M with $|N| = \alpha$ such that every countable set $\Gamma(x)$ of formulae in $L(M)$ realized in N is realized in M. Since T is ω-stable by lemma 4.2.4 T has a $\aleph_1$-saturated model of power α. By α-categoricity N is $\aleph_1$-saturated. Let $X \subseteq M$ with $|X| < \aleph_1$ and let $\Gamma(x)$ be a set of formulae in $L(X)$ consistent with $Th(M,a)_{a \in X}$. N is $\aleph_1$-saturated and so $\Gamma(x)$ is realized in N and hence in M. Hence M is $\aleph_1$-saturated. By

lemma 4.2.1 and its remark we have T is α-categorical for all $\alpha > \aleph_0$.

∎

For examples of α-categorical theories and applications of the above theorem we refer the reader to Chang and Keisler [73] pp 401, 402. We end off by mentioning that the above result can be generalized to uncountable theories as follows: If T is α-categorical for some $\alpha > |T|$ then T is α-categorical for all $\alpha > |T|$. (This was proved by Shelah in 1970.)

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